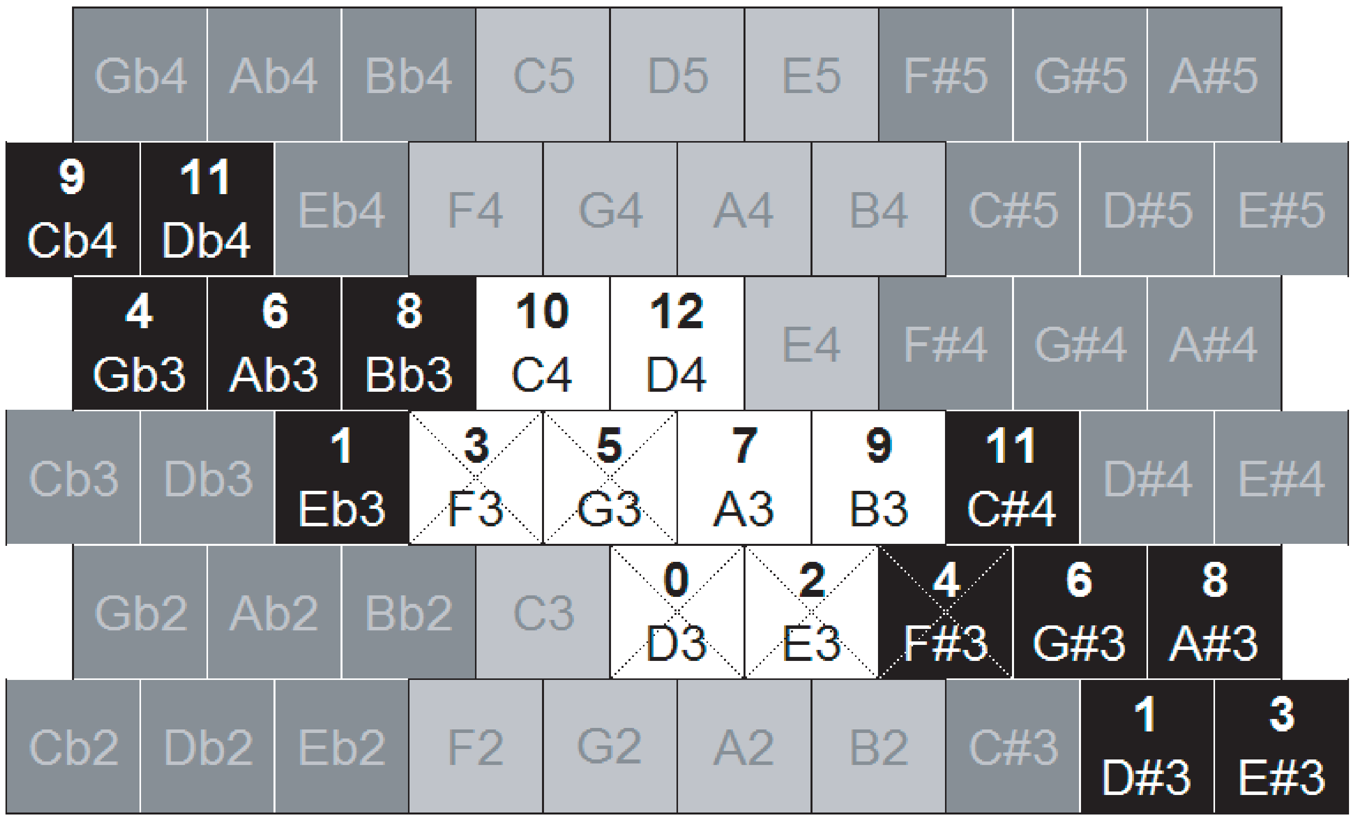
# Tuning Continua and Keyboard Layouts

by William Sethares, Andrew Milne and Jim Plamondon

A continuously parameterized family of tunings can be mapped to a button field so that the geometric shape of each musical interval is the same across all keys and throughout all tunings in the continuum.

# Transpositional Invariance




In this layout, intervals and chords are fingered the same in all keys.

## Idea of *Tuning Invariance*

Can we do the same kind of thing (have consistent fingering) across a range of tunings, instead of across all keys in a single tuning?

For example, can we arrange things so that (say) a 12-tet major chord, a 17-tet major chord, and a Pythagorean major chord all have the same fingering (while retaining transpositional invariance)?

When possible, there are several advantages:

- ease of learning new tunings
- ease of visualizing underlying structure of the music
- possibility of dynamically (re)tuning all sounded notes in real time throughout various tunings

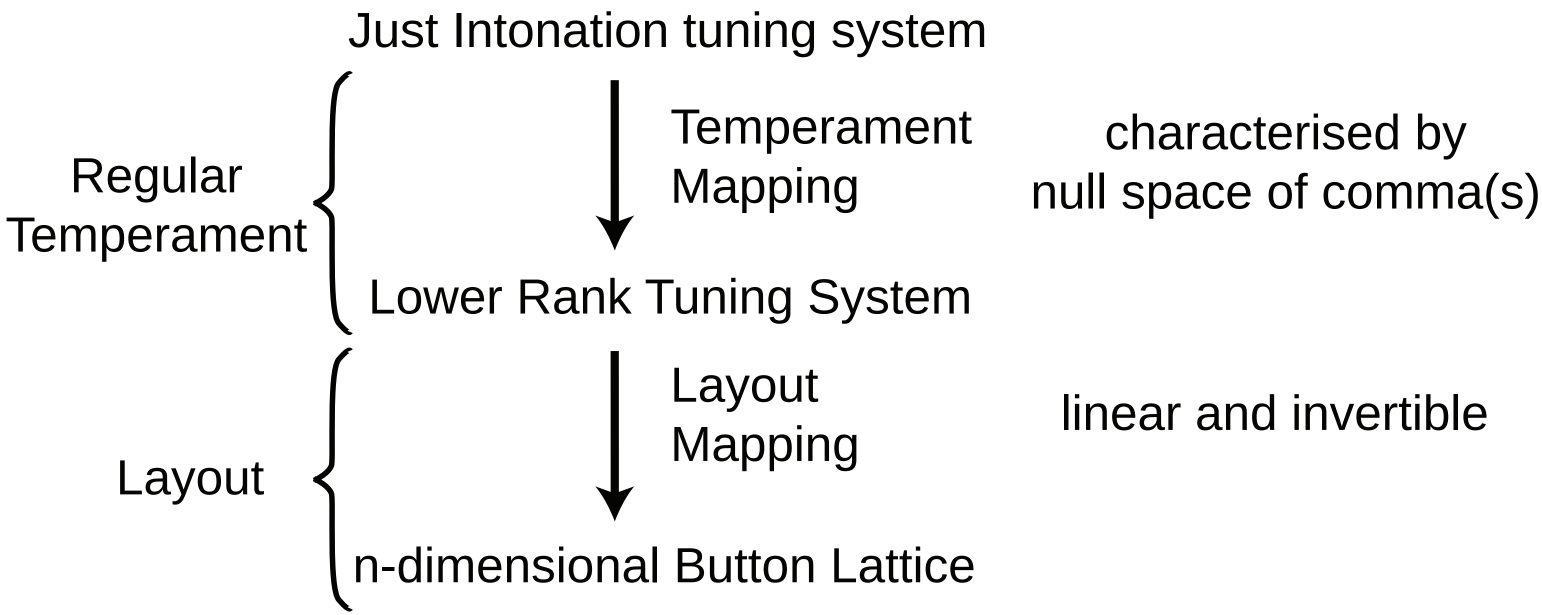
## Two Mappings and an Issue:

There are two mappings involved in the process: the first tempers from an arbitrary regular tuning to one that can be represented by a finite number of generators.

The second mapping is from the generators to the button field: translation invariance is shown to be equivalent to the linearity of this mapping, and consistent fingering occurs when the linear mapping is also invertible.

Issue: what does it mean to be the “same interval” or the “same chord” in multiple tunings?

# Two Mappings:



# Tempering by Commas: the General Case

Suppose a system $\mathcal{S}$ contains $p$ generators $g_1, g_2, \ldots, g_p$ where any element $s \in \mathcal{S}$ can be expressed as $g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ for integers $i_j$. The generators are *tempered* by $n < p$ *commas*, which means that the basis elements are replaced by nearby values

$$g_1 \to G_1, g_2 \to G_1, \ldots, g_p \to G_p$$

where

$$\begin{aligned} G_1^{c_{11}} G_2^{c_{12}} \cdots G_p^{c_{1p}} &= 1 \\ G_1^{c_{21}} G_2^{c_{22}} \cdots G_p^{c_{2p}} &= 1 \\ &\vdots \\ G_1^{c_{n1}} G_2^{c_{n2}} \cdots G_p^{c_{np}} &= 1. \end{aligned}$$

This set of constraints reduces the dimension from rank $p$ to rank $p-n = r$. Gather the coefficients of the commas into the matrix

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix}$$

and let $\mathcal{N}(\mathbf{C})$ be a basis for the null space of $\mathbf{C}$. Then the range space mapping $\mathbf{R} : \mathcal{Z}^p \rightarrow \mathcal{Z}^n$ has a basis defined by the transpose of $\mathcal{N}(\mathbf{C})$.

# Tempering by Commas: Example I

Consider the 5-limit primes defined by the generators 2, 3, 5, which are tempered to $2 \to G_1$, $3 \to G_2$, and $5 \to G_3$ by the *syntonic comma* $G_1^{-4} G_2^4 G_3^{-1} = 1$ and the *major diesis* $G_1^3 G_2^4 G_3^{-4} = 1$. Then

$$\mathbf{C} = \begin{pmatrix} -4 & 4 & -1 \\ 3 & 4 & -4 \end{pmatrix}$$

has null space $\mathcal{N}(\mathbf{C}) = (12, 19, 28)'$. Thus $\mathbf{R} = (12, 19, 28)$, and a typical element $2^{i_1} 3^{i_2} 5^{i_3}$ is tempered to $G_1^{i_1} G_2^{i_2} G_3^{i_3}$ and then mapped by $R$ to $12 i_1 + 19 i_2 + 28 i_3$. All three temperings can be written in terms of a single variable $\alpha$ as $G_1 = \alpha^{12}$, $G_2 = \alpha^{19}$, and $G_3 = \alpha^{28}$. If the choice is made to temper $G_1$ to 2 (to leave the octave unchanged) then $\alpha = \sqrt[12]{2}$ and the result is 12-tone equal temperament.

# Tempering by Commas: Example II

The 5-limit primes defined by the generators 2, 3, 5, may be tempered to $2 \to G_1$, $3 \to G_2$, and $5 \to G_3$ by the syntonic comma $G_1^{-4} G_2^4 G_3^{-1} = 1$. Then $\mathbf{C} = (-4, 4, -1)$ has null space spanned by the rows of

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

Typical elements $2^{i_1} 3^{i_2} 5^{i_3}$ are tempered to $G_1^{i_1} G_2^{i_2} G_3^{i_3}$ and then mapped via

$$\mathbf{R} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} i_1 + i_2 \\ -i_1 + 4i_3 \end{pmatrix}.$$

The tempered generators can be written in terms of two basis elements $\alpha$ and $\beta$ of the columns of $\mathbf{R}$ as $G_1 = \alpha\beta^{-1}$, $G_2 = \alpha$, and $G_3 = \beta^4$.

# $\alpha$-Reduced $\beta$-Chains

are generated by stacking integer powers of $\beta$ and then *reducing* (dividing or multiplying by $\alpha$) so that every term lies between 1 and $\alpha$. For any $i \in \mathbb{Z}$, the i$th$ note is

$$\beta^i \alpha^{-\lfloor i \log_\alpha(\beta) \rfloor}$$

where $\lfloor x \rfloor$ represents the largest integer less than or equal to $x$. $\alpha$-reduced $\beta$-chains define scales that repeat at intervals of $\alpha$; $\alpha = 2$, representing repetition at the octave, is the most common value.

## Valid Tuning Range

Any interval $s \in \mathcal{S}$ can be written in terms of the $p$ generators as $s = g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ where $i_j \in \mathbb{Z}$, or more concisely as the vector $\mathbf{s} = (i_1, i_2, \cdots i_p)$. A set of $n$ commas defines the temperament mapping $\mathbf{R}$. Consider a *privileged* set of intervals

$$1 = s_0 < s_1 < s_2 < \ldots < s_Q,$$

which can also be represented as the vectors $\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_Q$. Given any set of generators $\alpha_1, \alpha_2, \ldots, \alpha_{p-n}$ for the reduced rank tuning system, each interval $\mathbf{s}_i$ is tempered to $\mathbf{R}\mathbf{s}_i = \alpha_1^{j_1} \alpha_2^{j_2} \ldots \alpha_{p-n}^{j_{p-n}}$ where $j_k \in \mathbb{Z}$. The $\alpha$-generators define the specific tempered tuning and the coefficients $j_k$ specify the exact ratios of the privileged intervals within the temperament. The set of all generators $\alpha_i$ for which

$$1 = \mathbf{R}s_0 < \mathbf{R}s_1 < \mathbf{R}s_2 < \ldots < \mathbf{R}s_Q$$

holds is called the valid tuning range (VTR).

## Example: The Primary Consonances

Perhaps the most common example of a privileged set of intervals in 5-limit JI is the set of eight common practice consonant intervals

$$1, \frac{6}{5}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \frac{5}{3}, 2$$

which are familiar to musicians as the unison, just major and minor thirds, just perfect fifth, octave, and their octave inversions.

## Finding the VTR

The requirement that $\mathbf{R}s_{i+1} > \mathbf{R}s_i$ is identical to the requirement that $\mathbf{S}x > 0$ where

$$\mathbf{S} = \begin{pmatrix} \mathbf{s}_1 - \mathbf{s}_0 \\ \mathbf{s}_2 - \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{p-n} - \mathbf{s}_{p-n-1} \end{pmatrix} \text{ and } x = \begin{pmatrix} \log(\alpha_1) \\ \log(\alpha_2) \\ \vdots \\ \log(\alpha_{p-n}) \end{pmatrix}. \qquad (1)$$

(The inequality signifies an element-by-element operation.) This is the intersection of $p - n$ half-planes with boundaries that pass through the origin. The monotonicity assumption guarantees that the intersection is a nonempty cone radiating from the origin; this cone defines the VTR.

# Example: 5-Limit Syntonic Continuum

The primary consonances are mapped by $\mathbf{R}$ to $\begin{pmatrix} 0 & 5 & -6 & 2 & -1 & 7 & -4 & 1 \\ 0 & -3 & 4 & -1 & 1 & -4 & 3 & 0 \end{pmatrix}^{\mathsf{T}}$

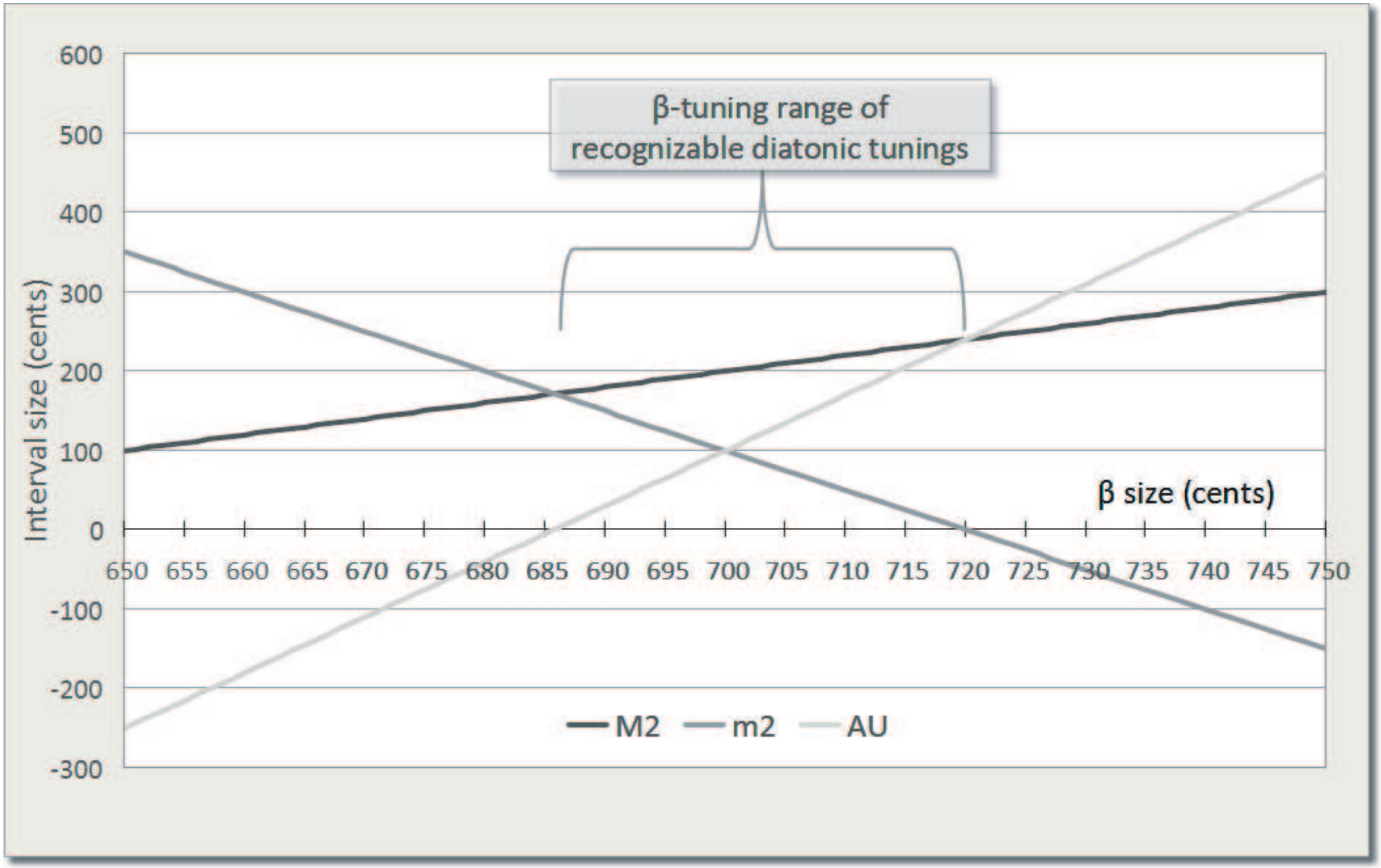
In terms of the generators $\alpha$ and $\beta$ this is

$\alpha^0\beta^0 < \alpha^5\beta^{-3} < \alpha^{-6}\beta^4 < \alpha^2\beta^{-1} < \alpha^{-1}\beta^1 < \alpha^7\beta^{-4} < \alpha^{-4}\beta^3 < \alpha^1\beta^0$

Rewriting this as a matrix gives

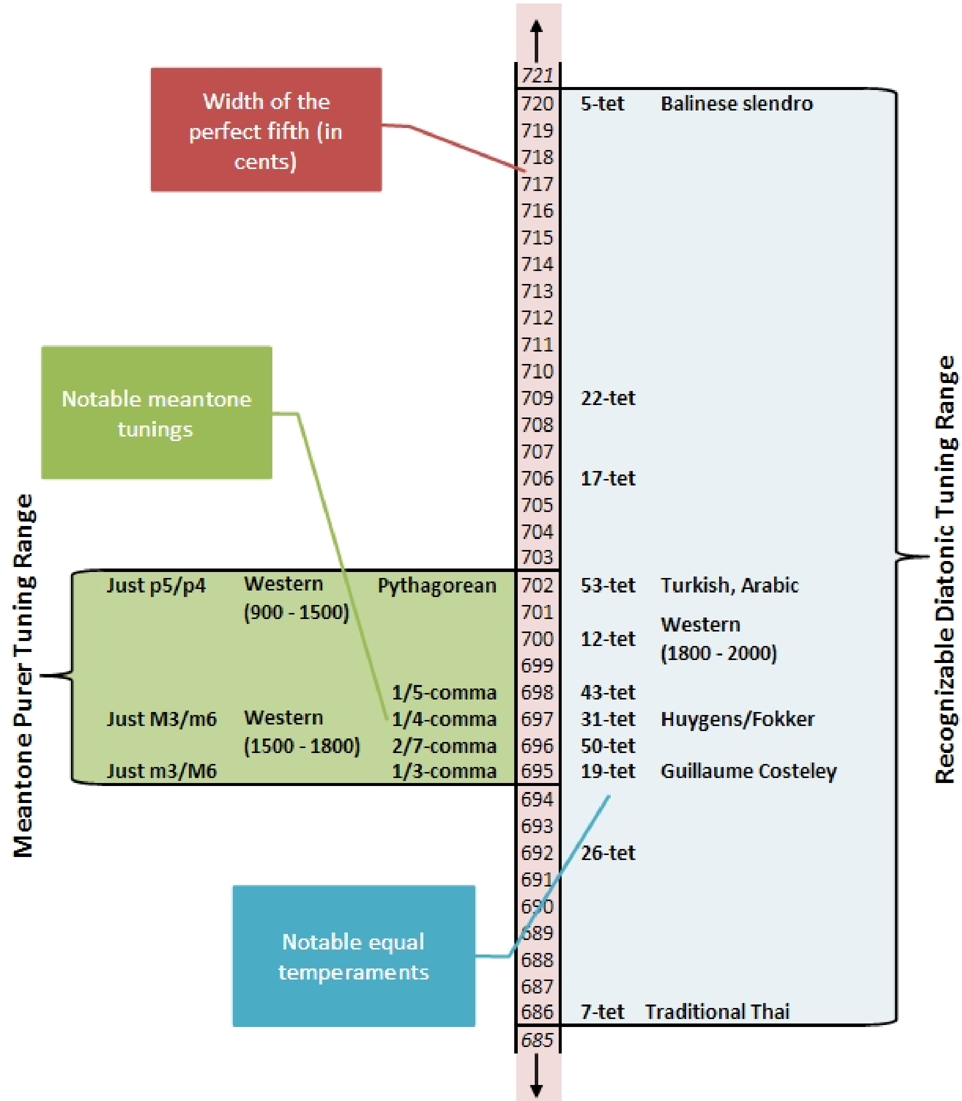
$$\begin{pmatrix} 5 & -11 & 8 & -3 & 8 & -11 & 5 \\ -3 & 7 & -5 & 2 & -5 & 7 & -3 \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This region is the cone bounded below by $x_2 = \frac{11}{7}x_1$ and bounded above by $x_2 = \frac{8}{5}x_1$, or $\alpha^{\frac{11}{7}} < \beta < \alpha^{\frac{8}{5}}$. For $\alpha = 2$, this covers the range between 7-edo and 5-edo. Outside this range, one or more of the privileged intervals changes fingering. This VTR range is identical to Blackwood's range of recognisable diatonic tunings, to the 12-note MOS scale generated by fifths.




Valid Tuning Ranges: With $\alpha = 2$, the size of major second (M2), minor second (m2), and augmented unison (AU) over a range of $\beta$.

The VTR for the syntonic continuum (given by the primary consonances) corresponds to Blackwood's range of "recognizable diatonic tunings" and to the 12-note MOS scale generated by fifths



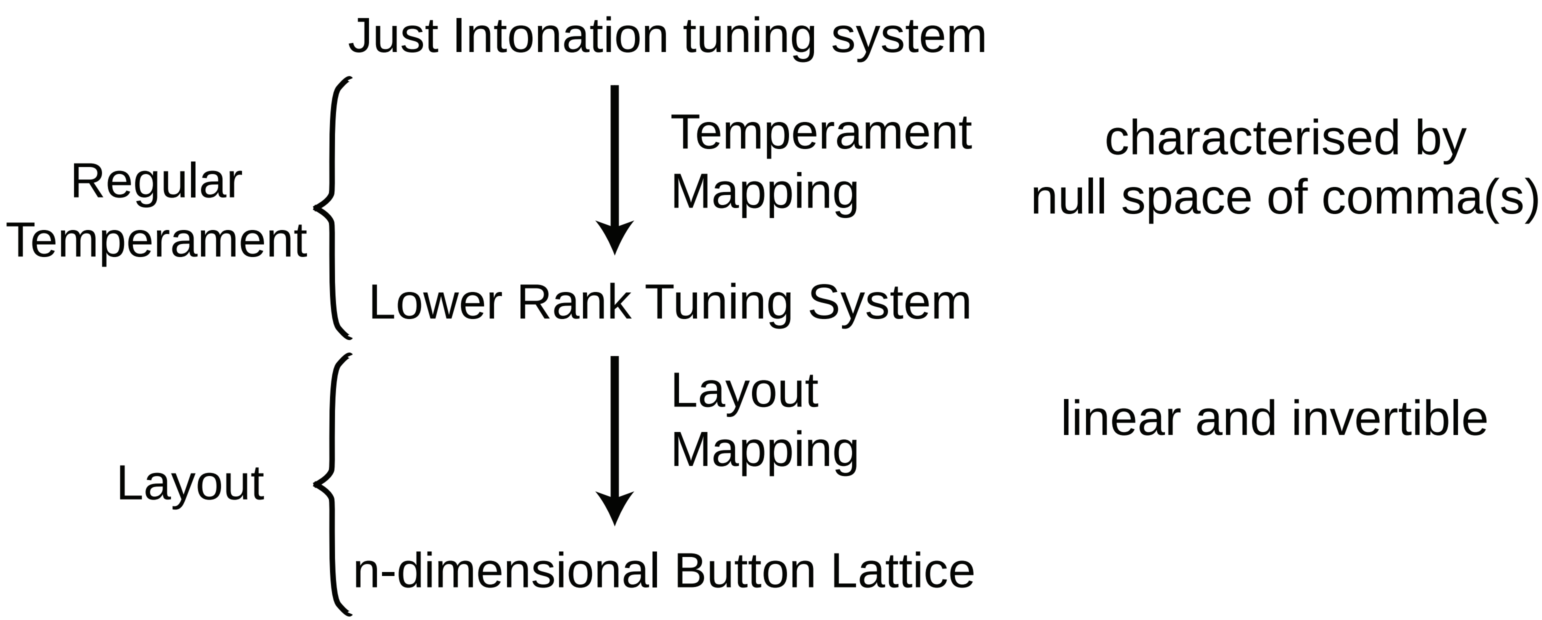
# VTRs are Easy:

A selection of temperaments with $2 \to \alpha$ and $1 < \beta < 2^{\frac{1}{2}}$. VTR values for the primary consonances are rounded to the nearest cent and the comma vectors presume that $2 \to G_1$, $3 \to G_2$, $5 \to G_3$.

| Common name | Negri | Porcupine | Hanson | Magic |
|---|---|---|---|---|
| Comma | $(-14, 3, 4)$ | $(1, -5, 3)$ | $(-6, -5, 6)$ | $(-10, -1, 5)$ |
| VTR (cents) | 120–150 | 150–171 | 300–327 | 360–400 |

| Common name | Würschmidt | Semisixths | Schismatic | Syntonic |
|---|---|---|---|---|
| Comma | $(17, 1, -8)$ | $(2, 9, -7)$ | $(-15, 8, 1)$ | $(-4, 4, -1)$ |
| VTR (cents) | 375–400 | 436–450 | 494–514 | 480–514 |

# Two Mappings:



# Layout Mappings

A layout mapping $\mathbf{L}$ is the physical embodiment of a mapping from a regular temperament to a button lattice.

(I) Linear layout mappings are transpositionaly invariant.

(II) Conversely: transposition invariance implies linearity of the layout mapping

$\mathbf{L}$ must be invertible, or else either some buttons would have no assigned note (or some notes would have no corresponding button). For fair comparisons, $\mathtt{det}(\mathbf{L}) = \pm 1$.

## Some Layout Matrices

Hexagonal: $\mathbf{B}_{\text{Hex}} = \begin{pmatrix} 1.07 & 0.54 \\ 0 & 0.93 \end{pmatrix}$

Square: $\mathbf{B}_{\text{Squ}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thummer: $\mathbf{B}_{\text{Thu}} = \begin{pmatrix} 1.25 & 0.62 \\ 0 & 0.80 \end{pmatrix}$

Wilson: $\mathbf{B}_{\text{Wil}} = \begin{pmatrix} 0.94 & 0.47 \\ 0.35 & 1.23 \end{pmatrix}$

$\mathbf{L}_1 = \begin{pmatrix} 0.54 & 1.07 \\ 0.93 & 0 \end{pmatrix}$

Wicki: $\mathbf{L}_{\text{Wic}} = \begin{pmatrix} 0 & 0.54 \\ 1.86 & 0.93 \end{pmatrix}$

$\mathbf{L}_2 = \begin{pmatrix} 2.69 & 1.61 \\ 0.93 & 0.93 \end{pmatrix}$

CBA-B: $\mathbf{L}_{\text{CBA-B}} = \begin{pmatrix} 3.76 & 2.15 \\ 2.79 & 1.86 \end{pmatrix}$

CBA-C: $\mathbf{L}_{\text{CBA-C}} = \begin{pmatrix} 3.76 & 2.15 \\ -2.79 & -1.86 \end{pmatrix}$

Fokker: $\mathbf{L}_{\text{Fok}} = \begin{pmatrix} 6.45 & 3.76 \\ 1.86 & 0.93 \end{pmatrix}$

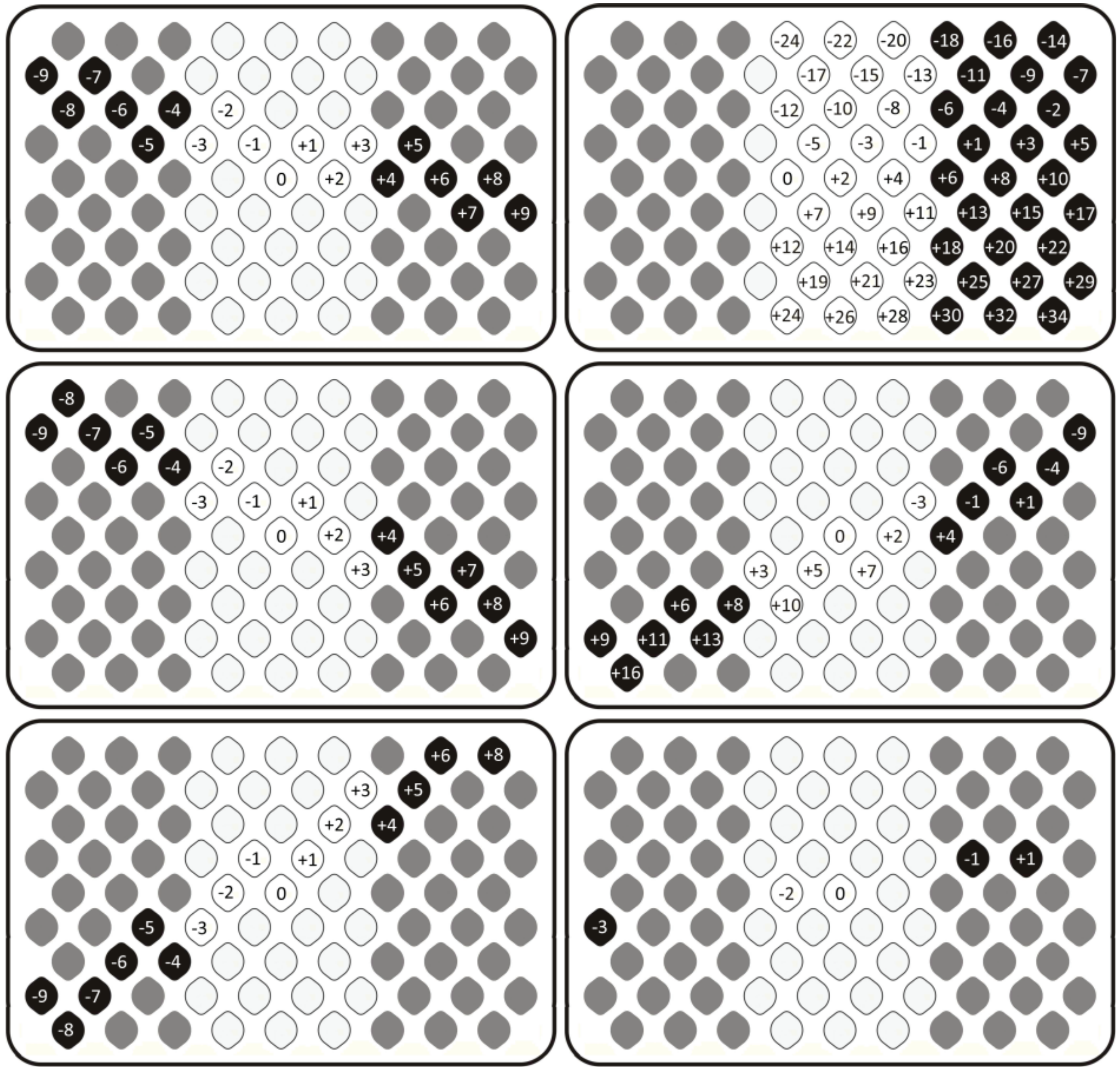
Bosanquet: $\mathbf{L}_{\text{Bos}} = \begin{pmatrix} 4.90 & 2.86 \\ 0 & 0.20 \end{pmatrix}$

Wilson: $\mathbf{L}_{\text{Wil}} = \begin{pmatrix} 5.66 & 3.30 \\ 0 & 0.18 \end{pmatrix}$

## The Layout Matrix Defines a Swathe

$$\mathbf{L} = (\psi \ \ \omega) = \begin{pmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{pmatrix}$$

As successive notes in an $\alpha$-reduced $\beta$-chain are laid onto a button-lattice they cut a *swathe* across it. The size of the swathe determines the microtonal and modulatory capabilities of the instrument; the number of $\alpha$-repetitions determines the overall pitch range of the instrument. The number of physical buttons on any given keyboard lattice limits the total number of intervals; the choice of layout $\mathbf{L}$ determines the trade-off.




Swathes produced by the Wicki layout (left) and the Fokker layout (right) for $z = \frac{7}{12}$ (first row), $z = \frac{2}{3}$ (second row), $z = \frac{1}{4}$ (third row).

# The Swathe

The vector position $\mathbf{v}_n$ of the $n$th note in a swathe can be expressed as a function of $\alpha$, $\beta$, $\psi$, and $\omega$ as

$$\mathbf{v}_n = n\omega - \lfloor nz \rfloor \psi \text{ where } z = \log_\alpha(\beta)$$

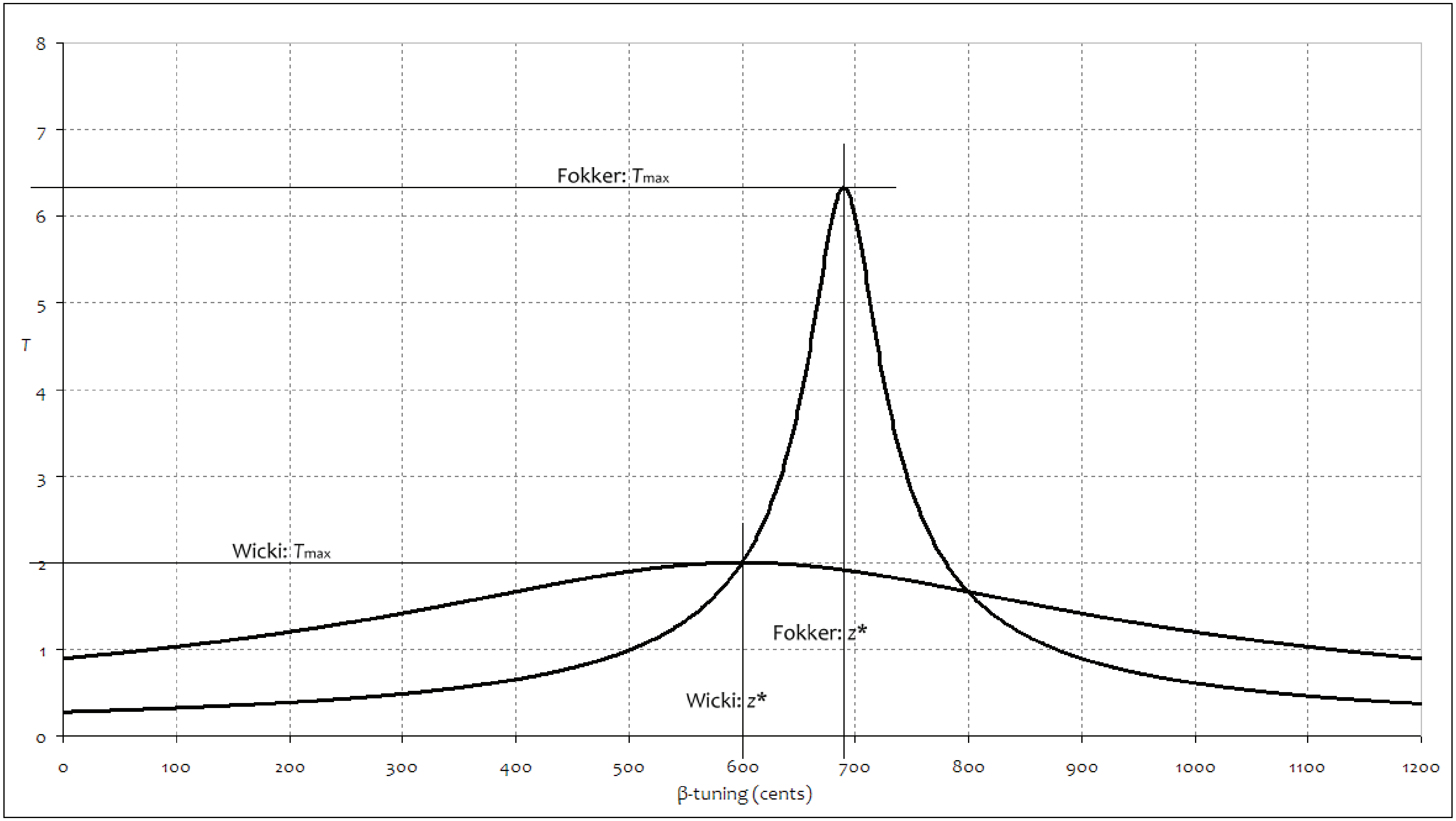
and $n \in \mathbb{Z}$. The slope

$$m = \frac{\omega_y - \psi_y z}{\omega_x - \psi_x z}$$

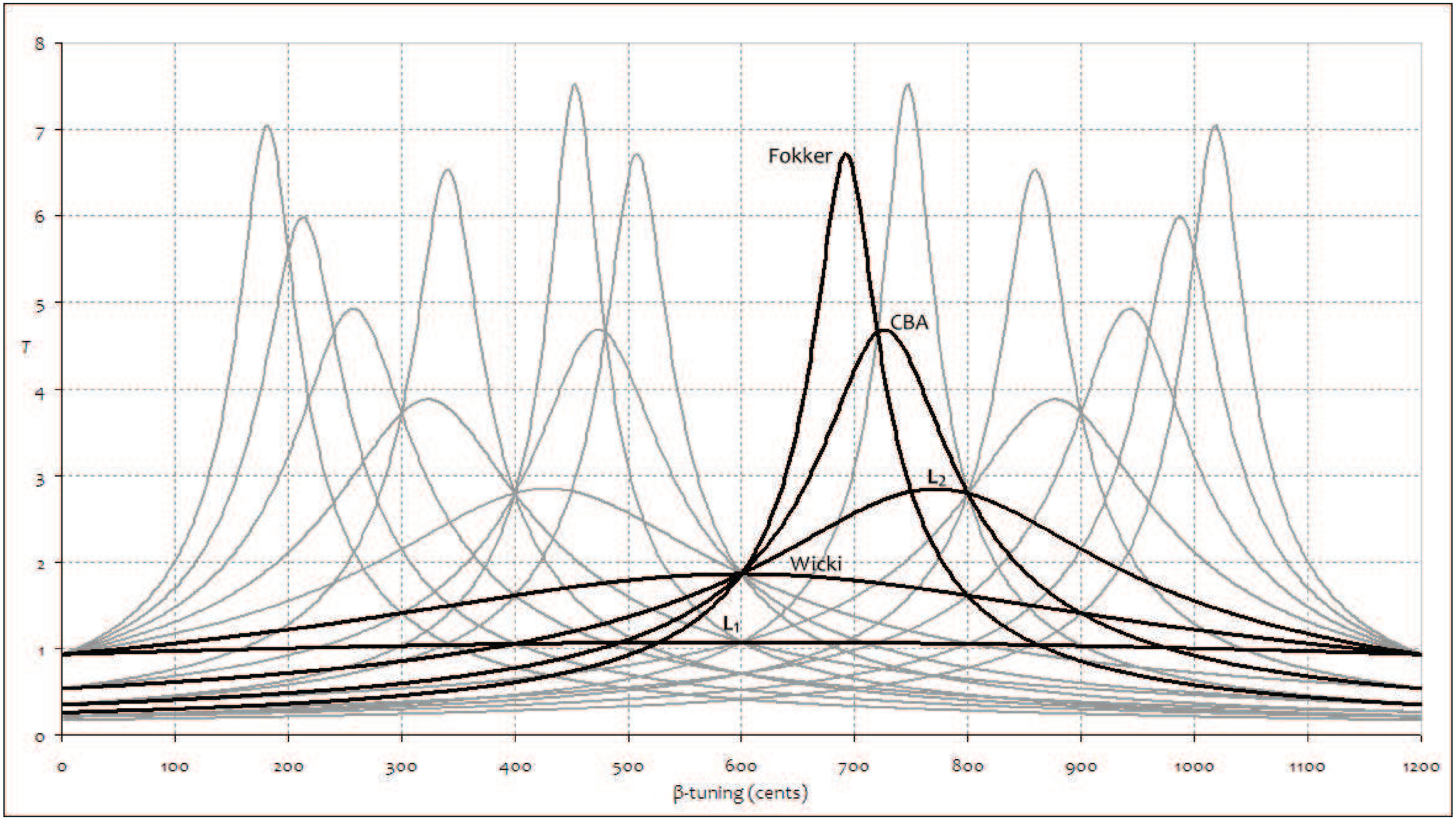
and thickness of the swathe are given by

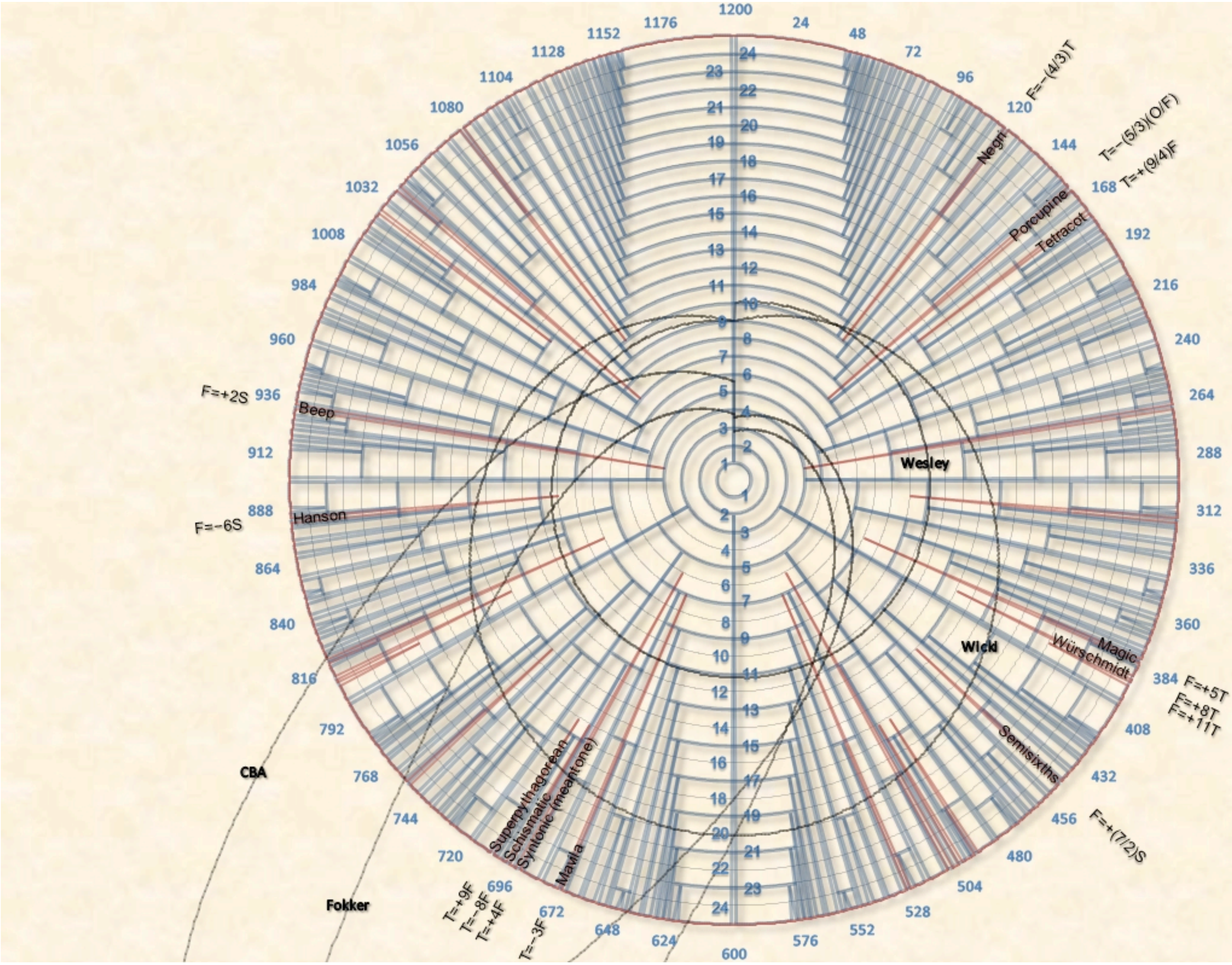
$$T = \frac{1}{\sqrt{(\omega_x - \psi_x z)^2 + (\omega_y - \psi_y z)^2}}.$$

Result: The higher the $T$, the narrower it is.



8
7
6
5
4
3
2
1
0
T
Fokker: $T_{max}$
Wicki: $T_{max}$
Fokker: $z^*$
Wicki: $z^*$
0
100
200
300
400
500
600
700
800
900
1000
1100
1200
β-tuning (cents)


T
β-tuning (cents)
Fokker
CBA
L2
Wicki
L1
0
1
2
3
4
5
6
7
8
0
100
200
300
400
500
600
700
800
900
1000
1100
1200

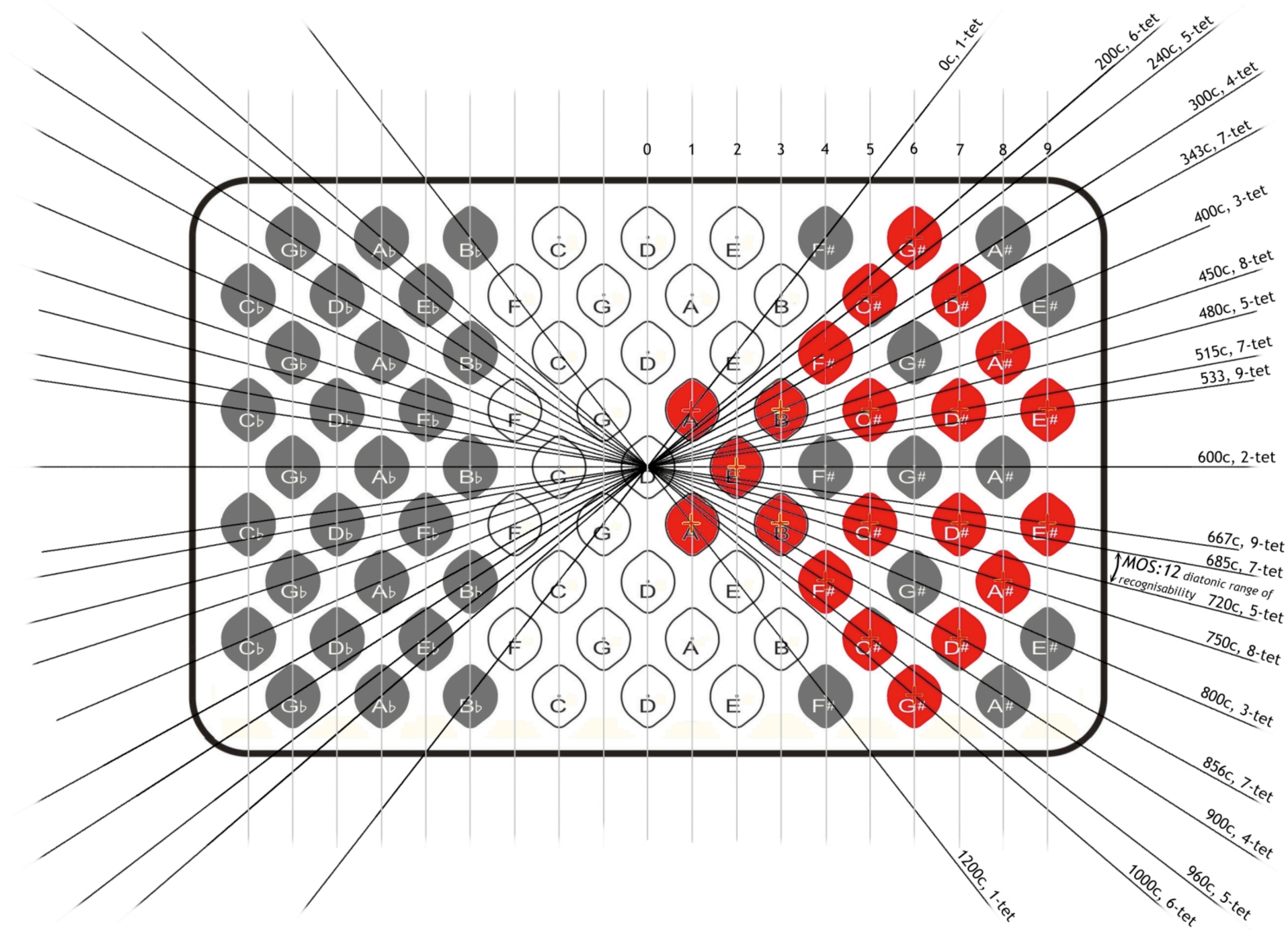

F=−(4/3)T
T=−(5/3)(O/F)
T=+(9/4)F
Negri
Porcupine
Tetracot
Wesley
Wicki
Magic
Würschmidt
F=+5T
F=+8T
F=+11T
Semisixths
F=+(7/2)S
F=+2S
Beep
Hanson
F=−6S
CBA
Fokker
T=+9F
T=−8F
T=+4F
T=−3F
Superpythagorean
Schismatic
Syntonic (meantone)
Mavila

## Monotonic pitch axis

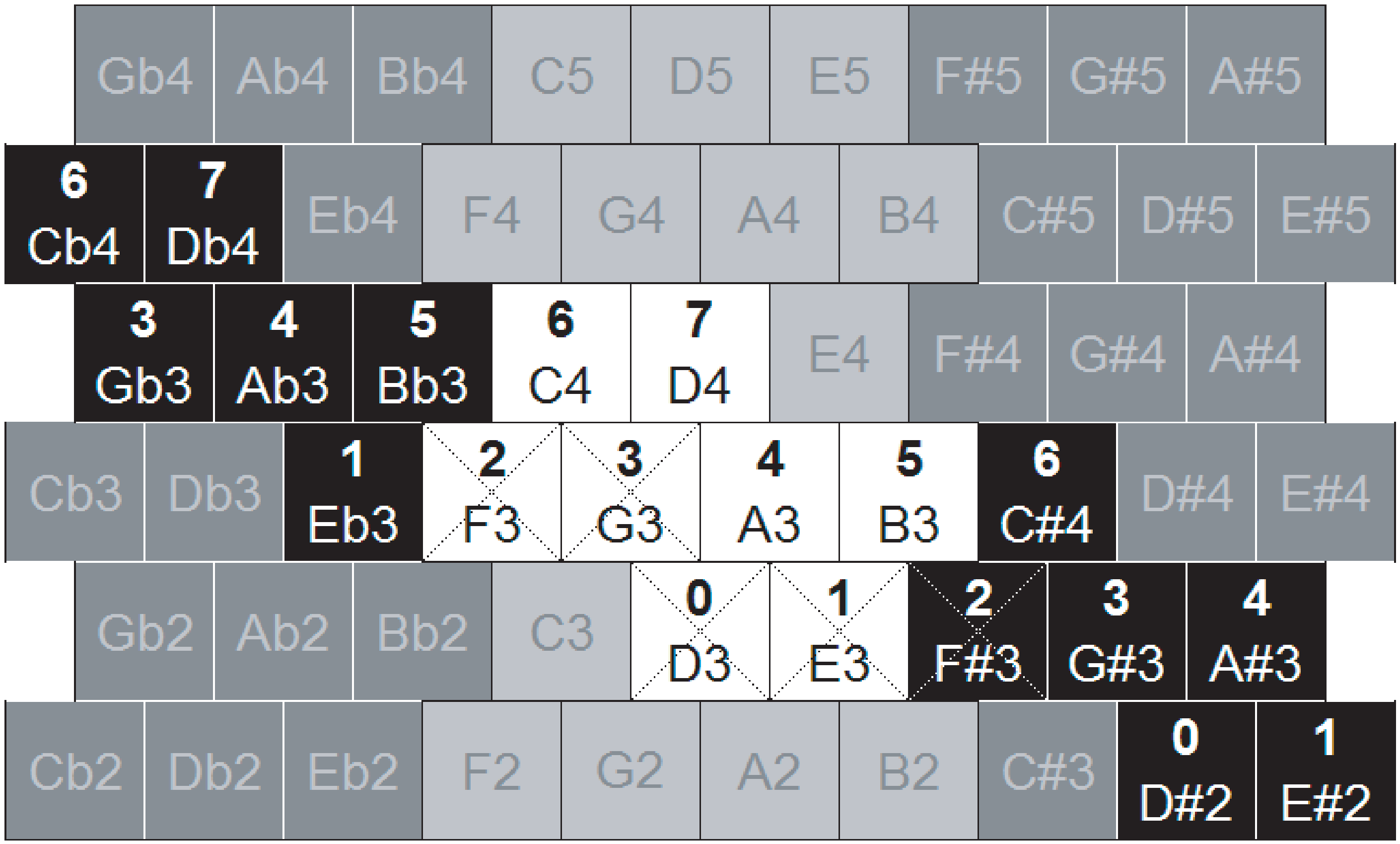
An *isotone* is a straight line drawn across a button-lattice that passes through the centres of buttons that produce equal pitch. The shortest distance of a button from an isotone is monotonically related to its pitch, so a line drawn at right angles to an isotone is called a *monotonic pitch axis*.

Result: An isotone has a slope equal to the swathe slope $m$.

Result: The shortest distance of a button from any given isotone is monotonically related to its pitch.

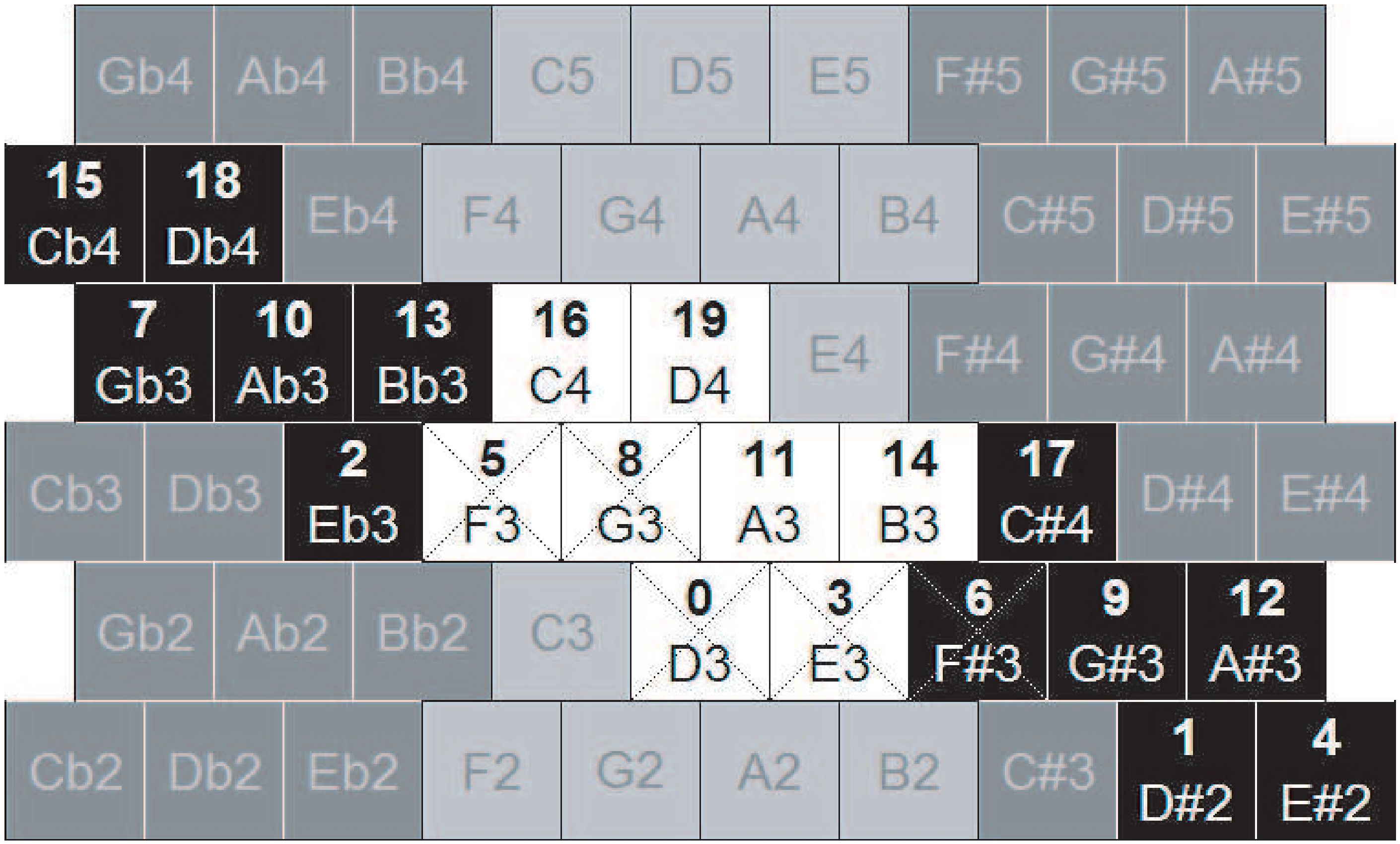

0 1 2 3 4 5 6 7 8 9
0c, 1-tet
200c, 6-tet
240c, 5-tet
300c, 4-tet
343c, 7-tet
400c, 3-tet
450c, 8-tet
480c, 5-tet
515c, 7-tet
533, 9-tet
600c, 2-tet
667c, 9-tet
685c, 7-tet
MOS:12 diatonic range of recognisability
720c, 5-tet
750c, 8-tet
800c, 3-tet
856c, 7-tet
900c, 4-tet
960c, 5-tet
1000c, 6-tet
1200c, 1-tet

# Through the Syntonic Tuning Continuum I



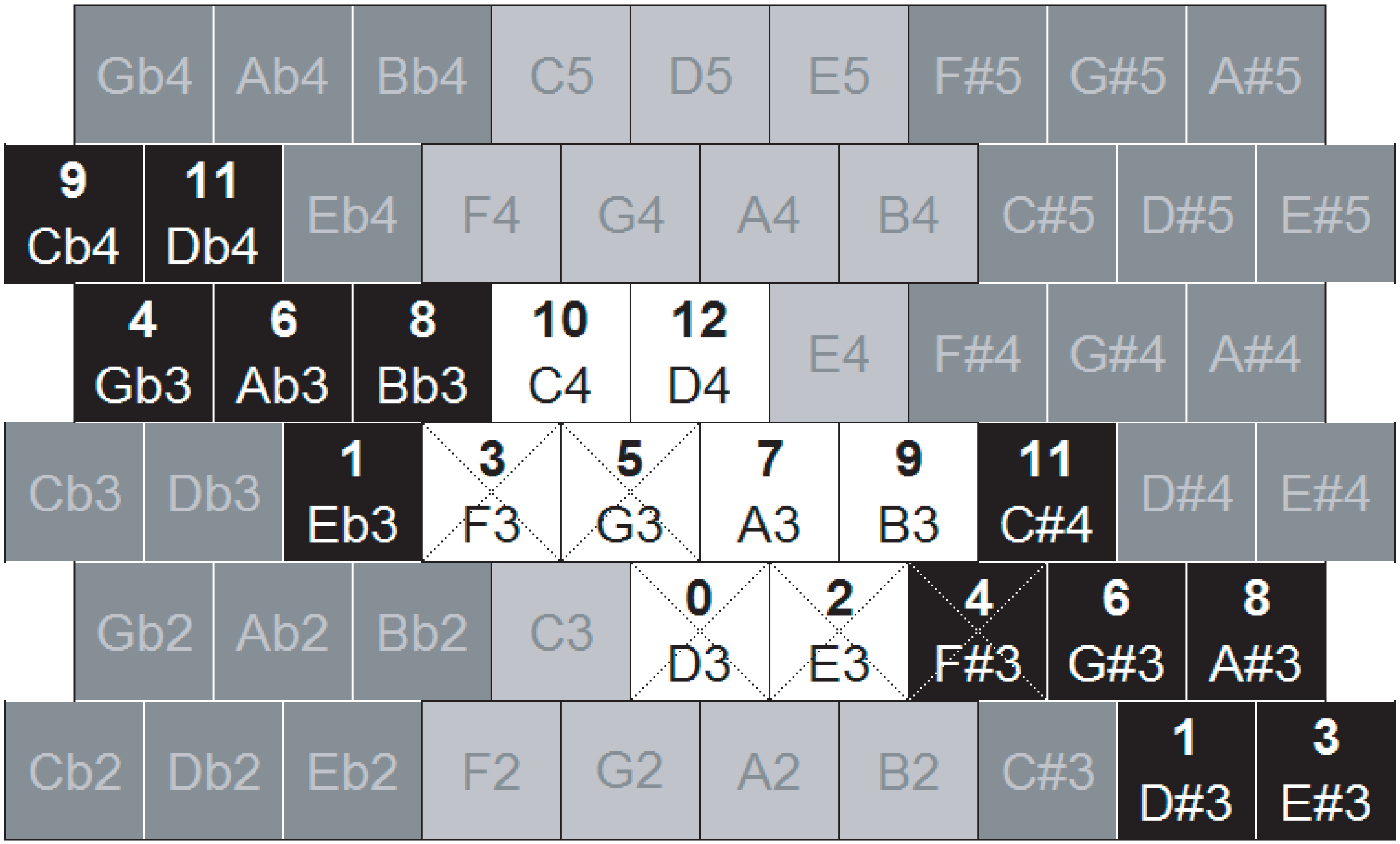

$$\beta = 2^{\frac{4}{7}} \text{ (7-tet)}$$

# Through the Syntonic Tuning Continuum II



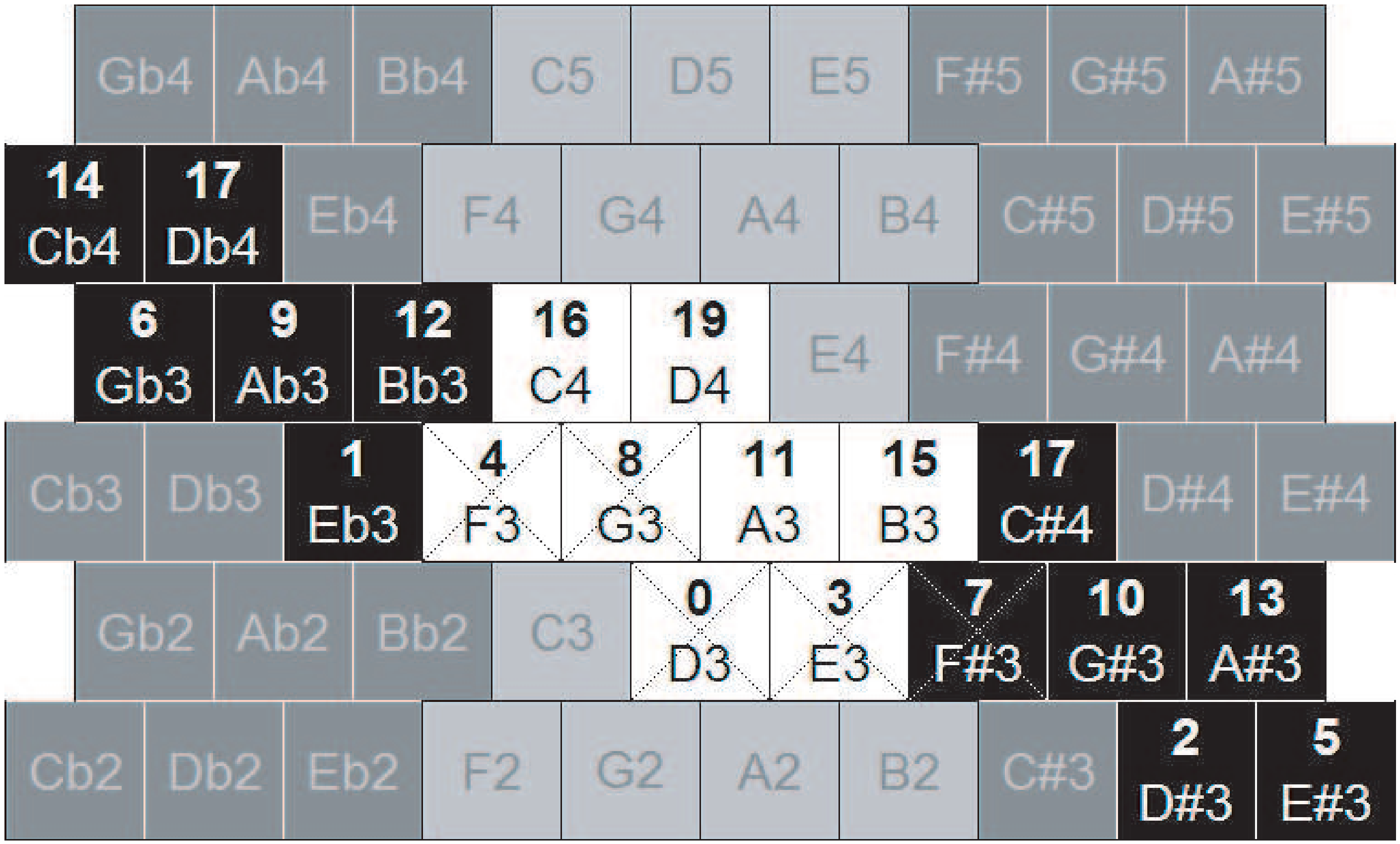

$$2^{\frac{4}{7}} < \beta < 2^{\frac{7}{12}}$$

# Through the Syntonic Tuning Continuum III



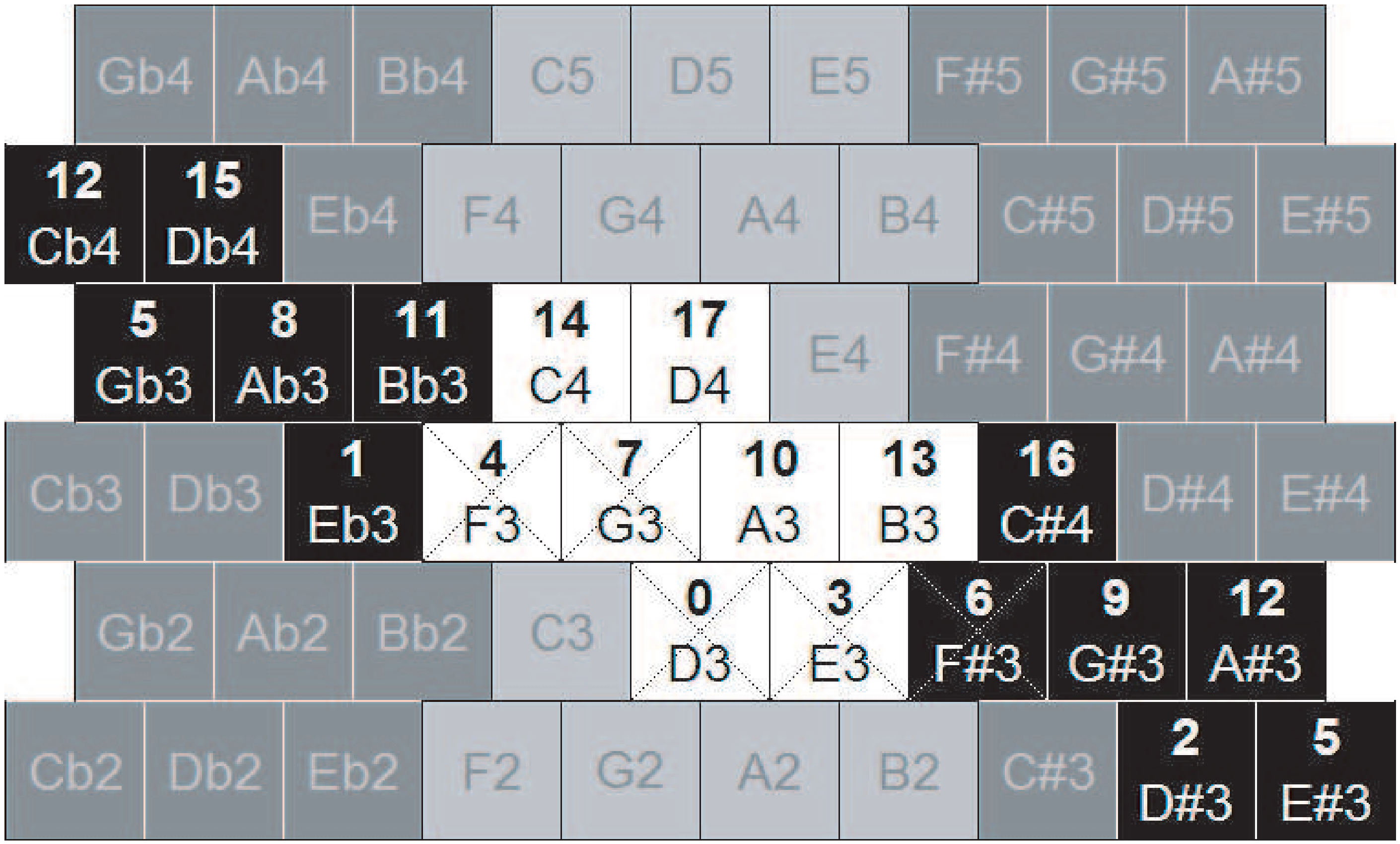

$$\beta = 2^{\frac{7}{12}} \text{ (12-tet)}$$

# Through the Syntonic Tuning Continuum IV



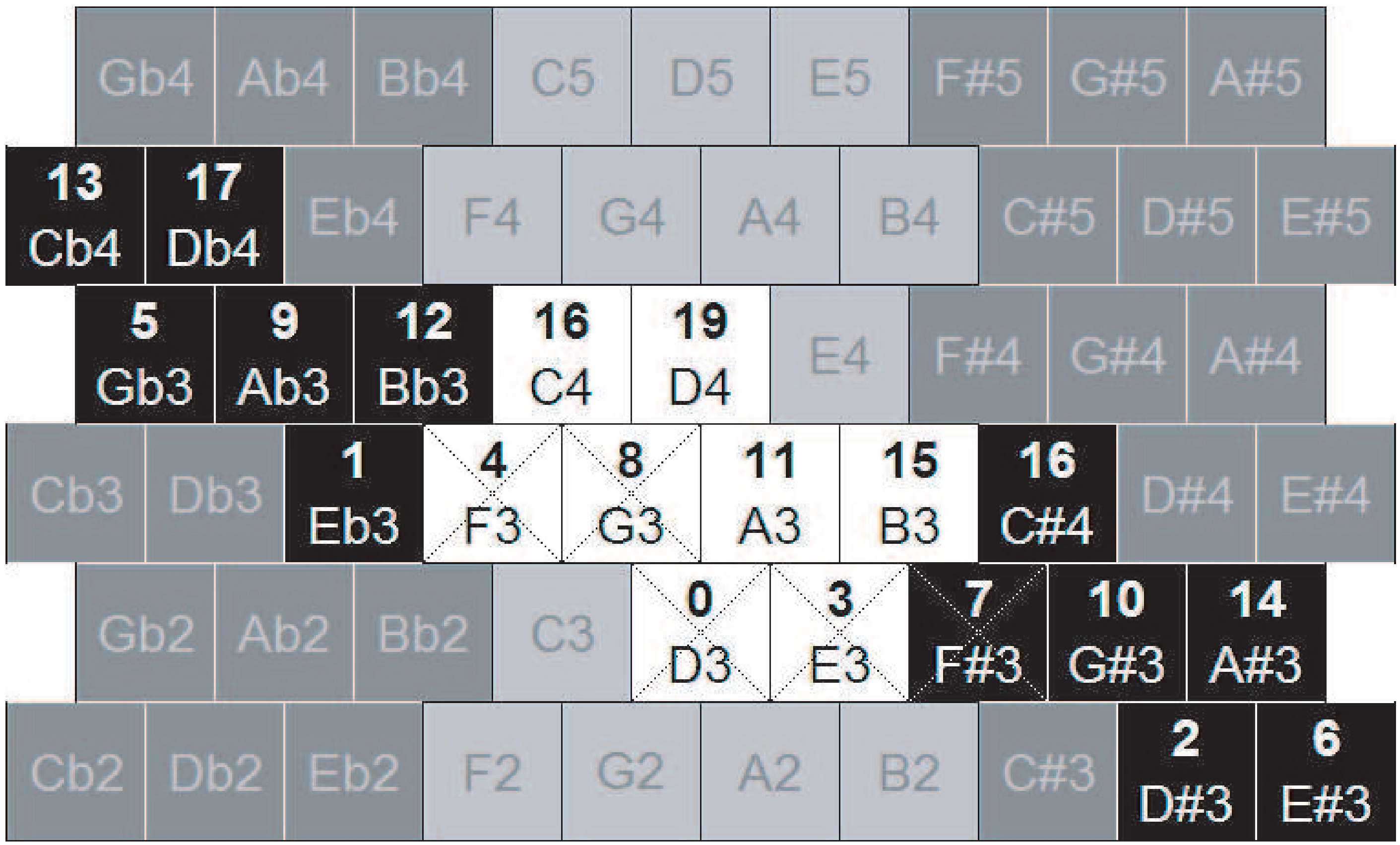

$$2^{\frac{7}{12}} < \beta < 2^{\frac{10}{17}}$$

# Through the Syntonic Tuning Continuum V



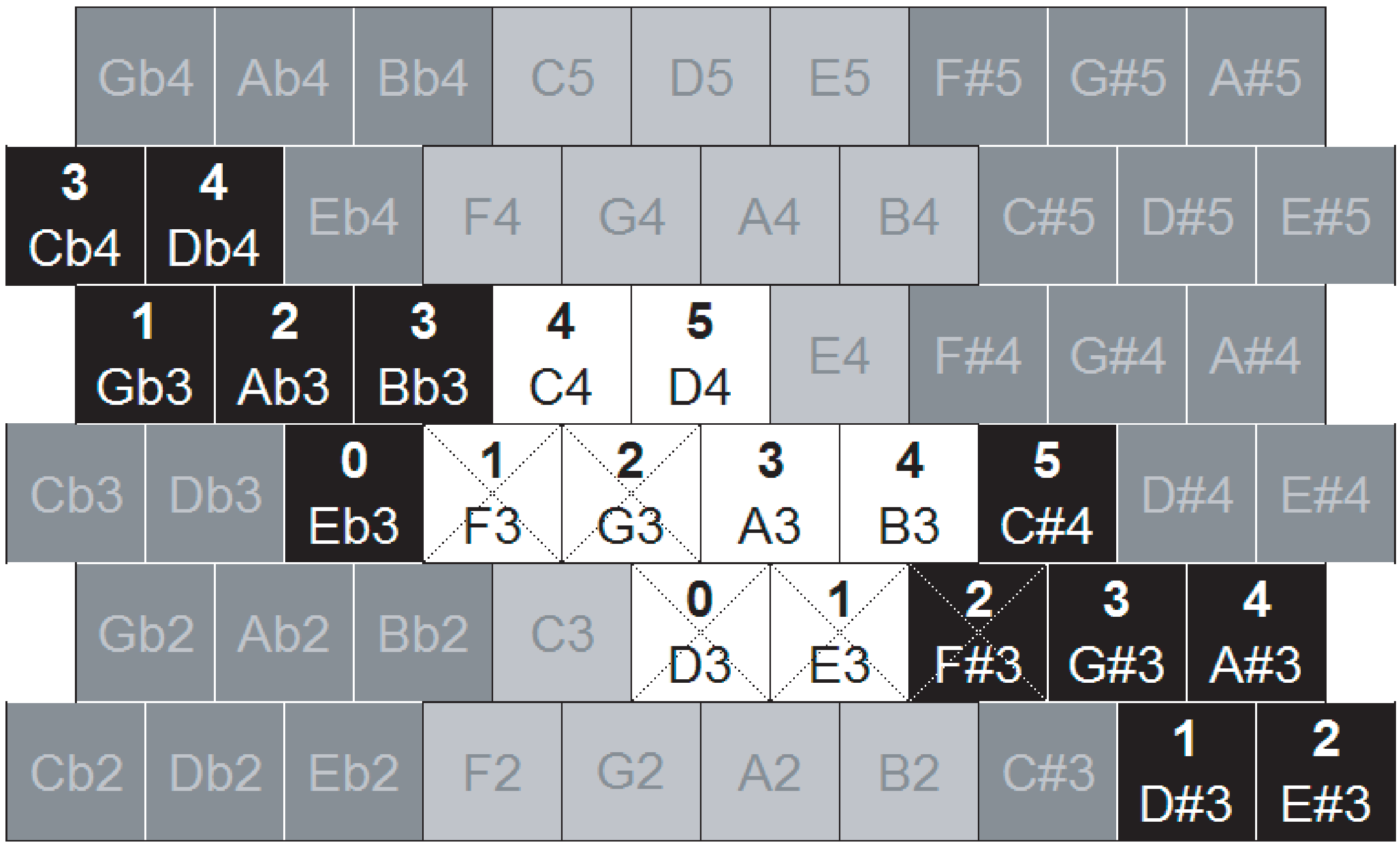

$$\beta = 2^{\frac{10}{17}} \text{ (17-tet)}$$

# Through the Diatonic Tuning Continuum VI




$$2^{\frac{10}{17}} < \beta < 2^{\frac{3}{5}}$$

# Through the Syntonic Tuning Continuum VII




$$\beta = 2^{\frac{3}{5}} \text{ (5-tet)}$$

...work in progress...

## But won't these weird temperings sound horribly dissonant?

In the same way that JI is related to the harmonic spectra (through the process of generating a dissonance curve with minima that lie at the desired scale steps), so the tempered intonations can be related to spectra with tempered partials.

The overtones of a sound can be matched to the temperaments in a straightforward way using the generators. The dissonance curves of these tempered harmonics have minima at the locations of the primary consonances of the related temperaments.

We can change the spectrum of the sounds along with the tunings!

Possible mappings (using 5-limit syntonic JI) for the harmonics as a function of the generators $\alpha$ and $\beta$:

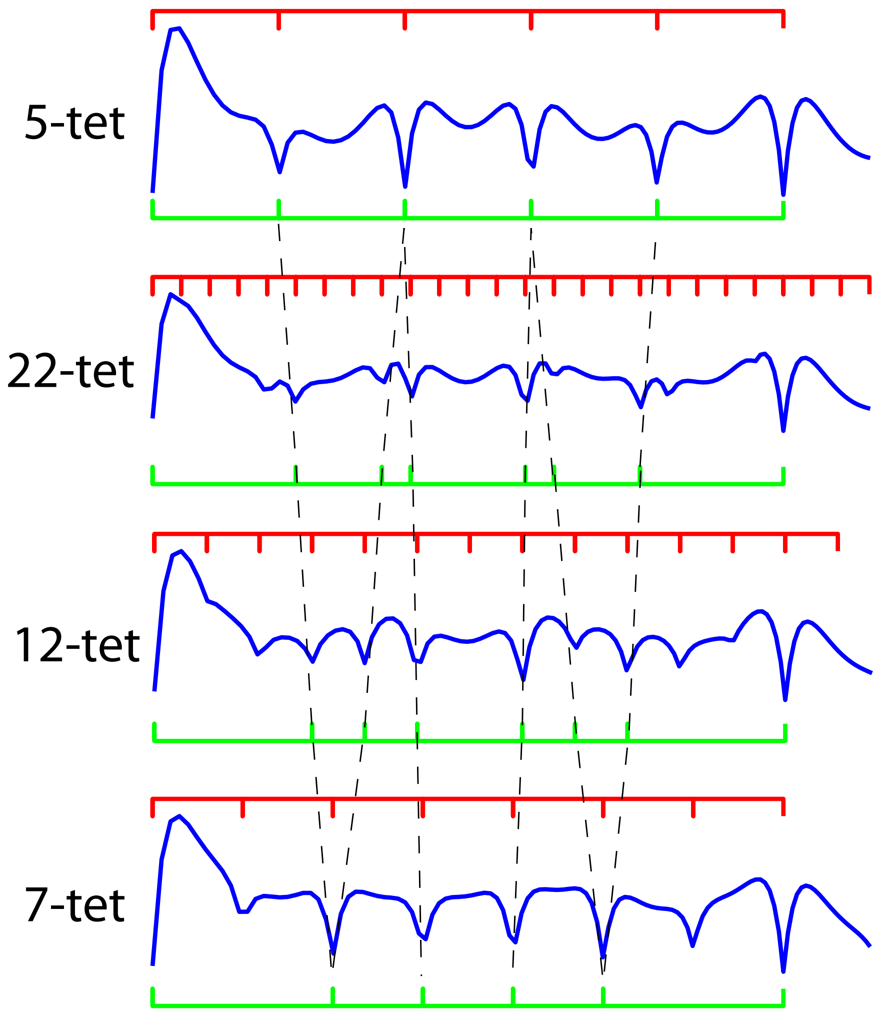
$O = \alpha$ (octave)

$F = \beta\alpha^{-1}$ (fifth)

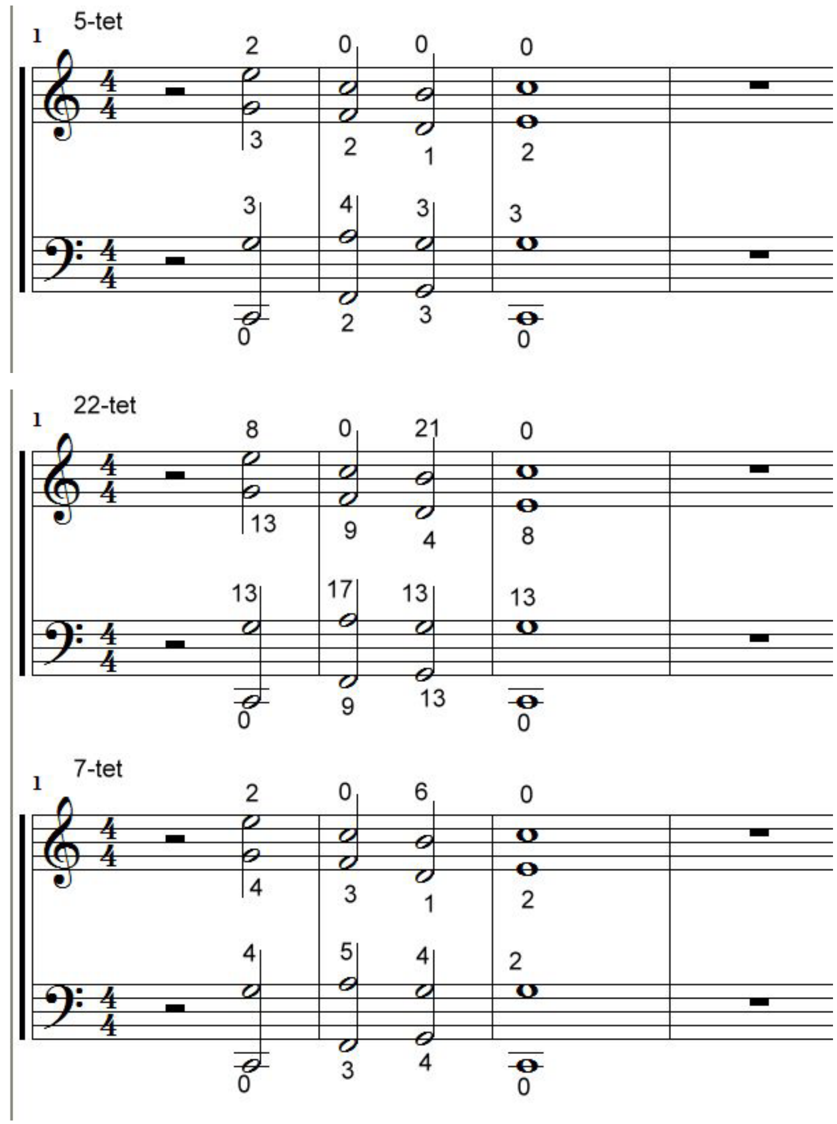
$T = \alpha^{-6}\beta^{4}$ (third)

| | 2, 3, 5 | jO, jF, jT | $O, F, T$ | $O, F, S$ |
|---|---|---|---|---|
| 1 | $1^1$ | 1 | 1 | 1 |
| 2 | $2^1$ | $\mathrm{jO}^1$ | $O^1$ | $O^1$ |
| 3 | $3^1$ | $\mathrm{jO}^1\ \mathrm{jF}^1$ | $O^1\ F^1$ | $O^1\ F^1$ |
| 4 | $2^2$ | $\mathrm{jO}^2$ | $O^2$ | $O^2$ |
| 5 | $5^1$ | $\mathrm{jO}^2\ \mathrm{jT}^1$ | $O^2\ T^1$ | $O^1\ F^1\ S^1$ |
| 6 | $2^1\ 3^1$ | $\mathrm{jO}^2\ \mathrm{jF}^1$ | $O^2\ F^1$ | $O^2\ F^1$ |
| 7 | $7^1$ | | | |
| 8 | $2^3$ | $\mathrm{jO}^3$ | $O^3$ | $O^3$ |
| 9 | $3^2$ | $\mathrm{jO}^2\ \mathrm{jF}^2$ | $O^2\ F^2$ | $O^2\ F^2$ |
| 10 | $2^1\ 5^1$ | $\mathrm{jO}^3\ \mathrm{jT}^1$ | $O^3\ T^1$ | $O^2\ F^1\ S^1$ |
| 11 | $11^1$ | | | |
| 12 | $2^2\ 3^1$ | $\mathrm{jO}^3\ \mathrm{jF}^1$ | $O^3\ F^1$ | $O^3\ F^1$ |
| 13 | $13^1$ | | | |
| 14 | $2^1\ 7^1$ | | | |
| 15 | $3^1\ 5^1$ | $\mathrm{jO}^3\ \mathrm{jF}^1\ \mathrm{jT}^1$ | $O^3\ F^1\ T^1$ | $O^2\ F^2\ S^1$ |
| 16 | $2^4$ | $\mathrm{jO}^4$ | $O^4$ | $O^4$ |

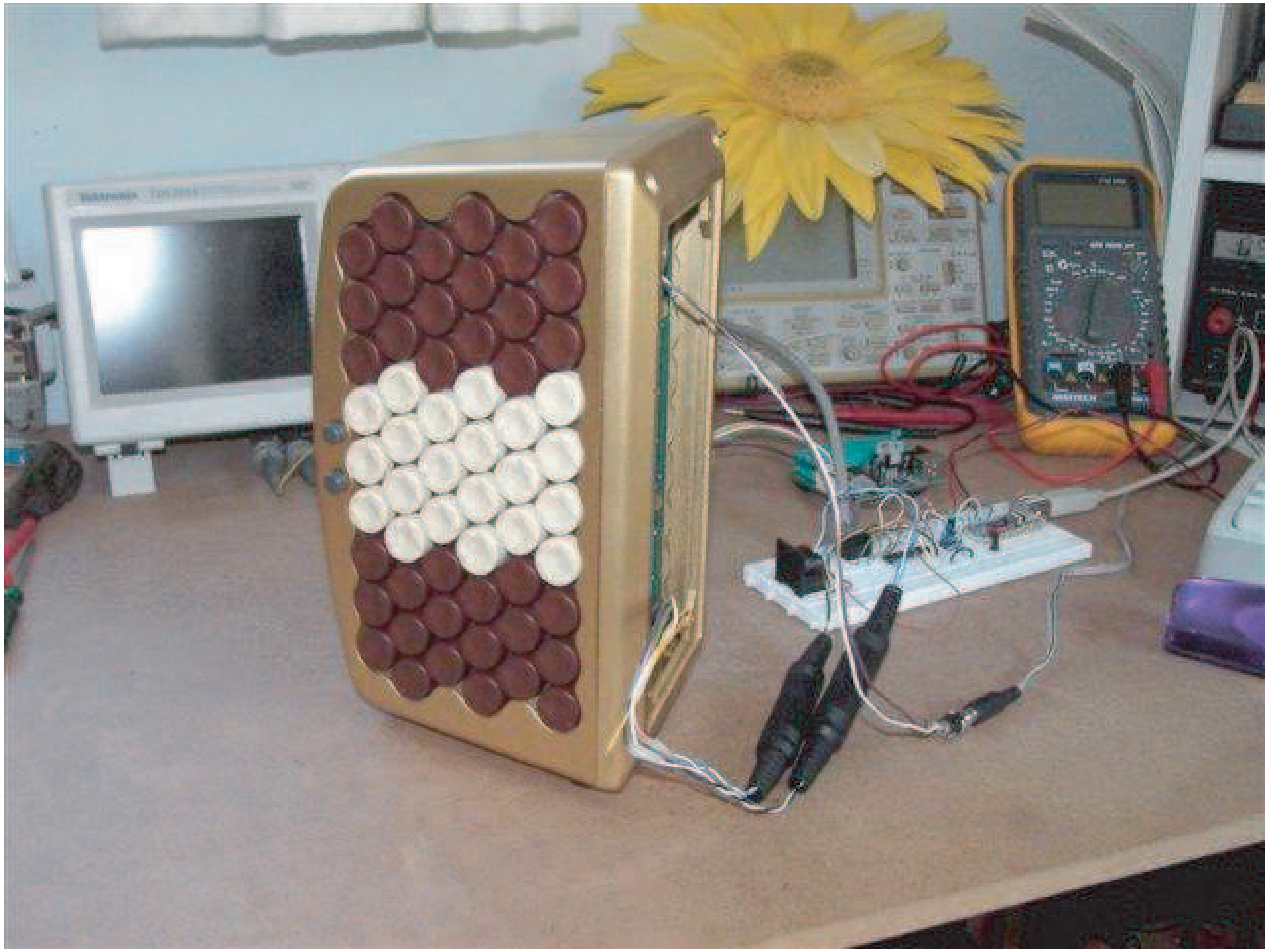
Dissonance curves for the various spectra created from the generators over the syntonic continuum have minima at the required primary consonances.



A I-IV-V-I progression annotated in several tunings throughout the syntonic continuum. Listen to the progressions with the tempered spectra.



# Prototype "Thummer" Keyboard



## Summary

It is possible to design keyboards capable of smoothly moving among a continuum of tunings, retaining the same fingerings in all keys over the continuum. This uses a parameterization based on commas.

The Valid Tuning Range can be easily calculated in terms of a set of privileged intervals (e.g., the primary consonances)

Linear Layout maps can be understood in terms of properties of the swathe (slope and thickness) and of a monotonic pitch axis.

It is also possible to modify the spectra of sounds so that a degree of consonance can be maintained throughout the continuum.