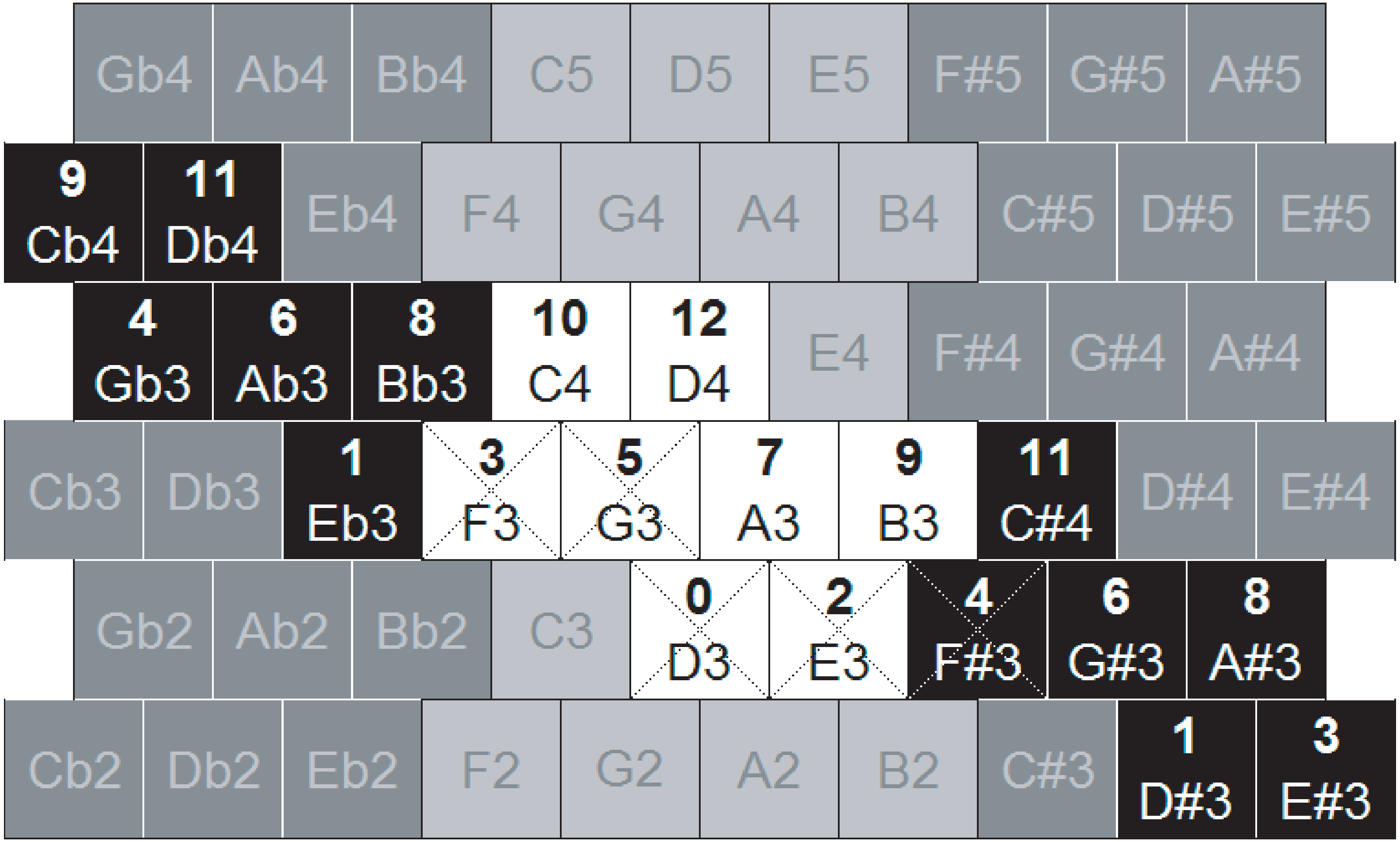
Consistent Fingerings for a Continuum of Syntonic Tunings

by William Sethares, Andrew Milne and Jim Plamondon

A continuous parameter generates a continuum of syntonic tunings that can be mapped to a button field so that the geometric shape of each musical interval is the same within a key, across all keys, and throughout all tunings in the continuum.

Transpositional Invariance



In this layout, intervals and chords are fingered the same in all keys.

Idea of *Tuning Invariance*

Can we do the same kind of thing (have consistent fingering) across a range of tunings, instead of across all keys in a single tuning?

For example, can we arrange things so that (say) a 12-tet major chord, a 17-tet major chord, and a Pythagorean major chord all have the same fingering (while retaining transpositional invariance)?

When possible, there are several advantages:

- ease of learning new tunings
- ease of visualizing underlying structure of the music
- possibility of dynamically (re)tuning all sounded notes in real time throughout various tunings

Two Mappings and an Issue:

There are two mappings involved in the process: the first tempers from an arbitrary regular tuning to one that can be represented by a finite number of generators.

The second mapping is from the generators to the button field: translation invariance is shown to be equivalent to the linearity of this mapping, and consistent fingering occurs when the linear mapping is also invertible.

Issue: what does it mean to be the “same interval” or the “same chord” in multiple tunings?

Two Mappings:

Just Intonation tuning system

Regular Temperament

Temperament Mapping

characterized by null space of comma(s)

Lower Rank Tuning System

Layout

Layout Mapping

linear, determinant=1 integer valued

n-dimensional Button Lattice

Tempering by Commas: the General Case

Suppose a system $\mathcal{S}$ contains p generators $g_1, g_2, \ldots, g_p$ where any element $s \in \mathcal{S}$ can be expressed as $g_1^{i_1} g_2^{i_2} \cdots g_p^{i_p}$ for integers i_j. The generators are *tempered* by $n < p$ *commas*, which means that the basis elements are replaced by nearby values

$$g_1 \to G_1, g_2 \to G_1, \ldots, g_p \to G_p$$

where

$$\begin{aligned} G_1^{c_{11}} G_2^{c_{12}} \cdots G_p^{c_{1p}} &= 1 \\ G_1^{c_{21}} G_2^{c_{22}} \cdots G_p^{c_{2p}} &= 1 \\ &\vdots \\ G_1^{c_{n1}} G_2^{c_{n2}} \cdots G_p^{c_{np}} &= 1. \end{aligned}$$

This set of constraints reduces the dimension from rank p to rank

$p - n = r$. Gather the coefficients of the commas into the matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{np} \end{pmatrix}$$

and let $\mathcal{N}(C)$ be a basis for the null space of C. Then the range space mapping $R : \mathcal{Z}^p \to \mathcal{Z}^n$ has a basis defined by the transpose of $\mathcal{N}(C)$.

Tempering by Commas: Example I

Consider the 5-limit primes defined by the generators 2, 3, 5, which are tempered to $2 \to G_1$, $3 \to G_2$, and $5 \to G_3$ by the *syntonic comma* $G_1^{-4} G_2^4 G_3^{-1} = 1$ and the *major diesis* $G_1^3 G_2^4 G_3^{-4} = 1$. Then

$$C = \begin{pmatrix} -4 & 4 & -1 \\ 3 & 4 & -4 \end{pmatrix}$$

has null space $\mathcal{N}(C) = (12, 19, 28)'$. Thus $R = (12, 19, 28)$, and a typical element $2^{i_1} 3^{i_2} 5^{i_3}$ is tempered to $G_1^{i_1} G_2^{i_2} G_3^{i_3}$ and then mapped by R to $12i_1 + 19i_2 + 28i_3$. All three temperings can be written in terms of a single variable α as $G_1 = \alpha^{12}$, $G_2 = \alpha^{19}$, and $G_3 = \alpha^{28}$. If the choice is made to temper G_1 to 2 (to leave the octave unchanged) then $\alpha = \sqrt[12]{2}$ and the result is 12-tone equal temperament.

Tempering by Commas: Example II

The 5-limit primes defined by the generators 2, 3, 5, may be tempered to $2 \to G_1$, $3 \to G_2$, and $5 \to G_3$ by the syntonic comma $G_1^{-4} G_2^4 G_3^{-1} = 1$. Then $C = (-4, 4, -1)$ has null space spanned by the rows of

$$R = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 4 \end{pmatrix}.$$

Typical elements $2^{i_1} 3^{i_2} 5^{i_3}$ are tempered to $G_1^{i_1} G_2^{i_2} G_3^{i_3}$ and then mapped via

$$R \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \begin{pmatrix} i_1 + i_2 \\ -i_1 + 4i_3 \end{pmatrix}.$$

The tempered generators can be written in terms of two basis elements α and β of the columns of R as $G_1 = \alpha\beta^{-1}$, $G_2 = \alpha$, and $G_3 = \beta^4$.

Two Mappings:

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Layout Mappings

A layout mapping M is the physical embodiment of a mapping from a regular temperament to an integer-valued button lattice.

(I) Linear layout mappings are transpositionaly invariant.

(II) Conversely: transposition invariance implies linearity of the layout mapping

The elements of M must be integers (or else some intervals will be mapped to locations without buttons), M must be invertible (i.e., $\det(M) \neq 0$, or else either some buttons would have no assigned note or some notes would have no corresponding button) and $\det(M) = \pm 1$ (or else the inverse will not be integer valued).

Two Ways of Identifying Intervals

(1) Harmonically: occurs for harmonic intervals (formed from simultaneously sounded notes). It presumes a set of perceptual and/or cognitive “landmarks” against which sounded intervals can be mentally compared and identified. For example: low ratio just intervals, perceived as consonant, may provide such landmarks for harmonic timbres.

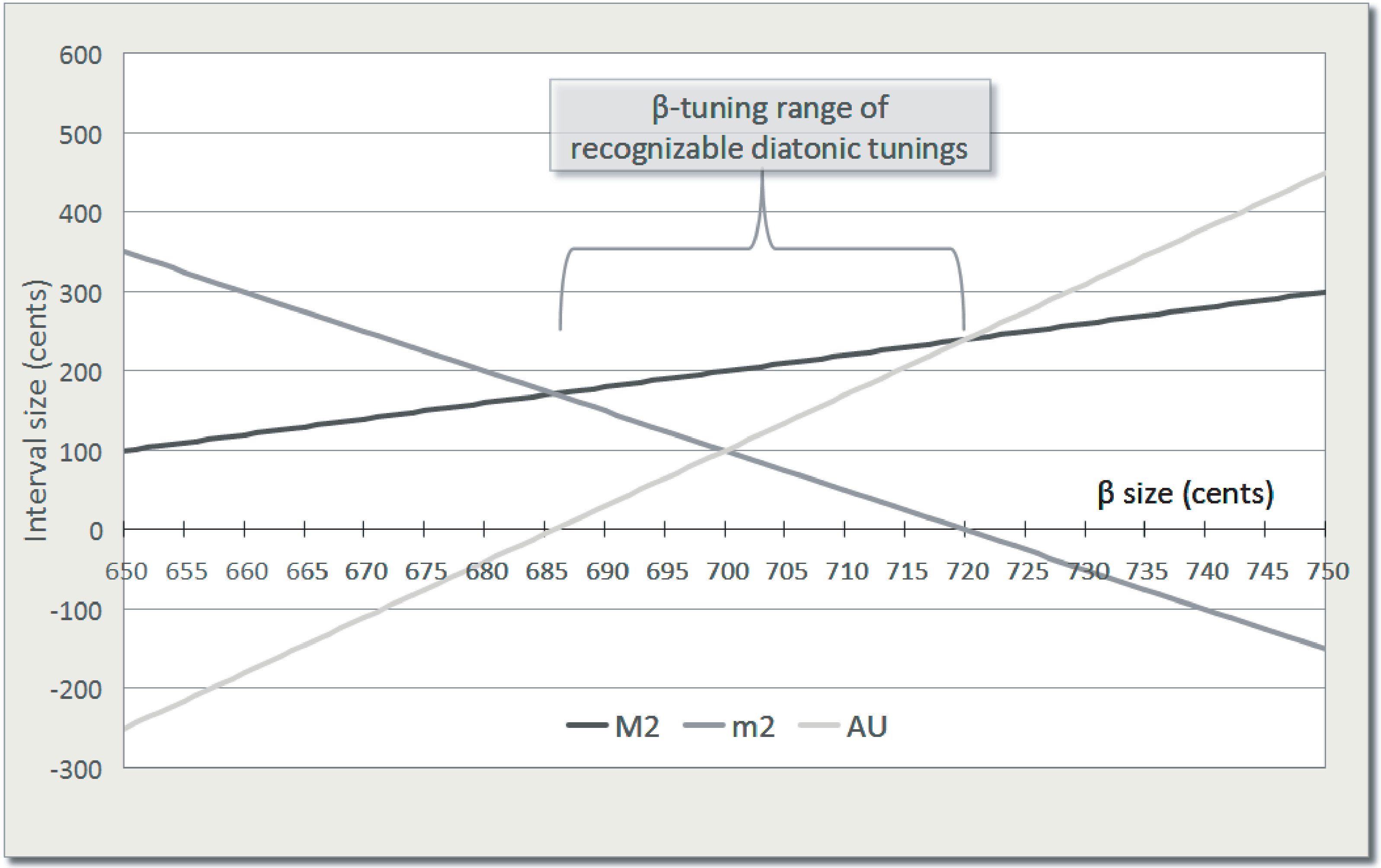
(2) Melodically: intervals formed from successively sounded notes presume that when an interval is played as part of a conventionalized scale, it is identified by the *number of scale notes* (or steps) that it spans.

Tuning Ranges for Invariant Identification

The valid tuning range for a Moment of Symmetry scale [Clough, Grady, Wilson] with m notes and generators α and β can be described explicitly. Let L be the largest value of $\alpha^{\frac{p}{q}}$ that is less than β, where p and q are coprime integers with $q < m$. Let U be the smallest value of $\alpha^{\frac{r}{s}}$ that is greater than β where r and s are coprime integers with $s < m$. Then the generator must lie in the region $L < \beta < U$.

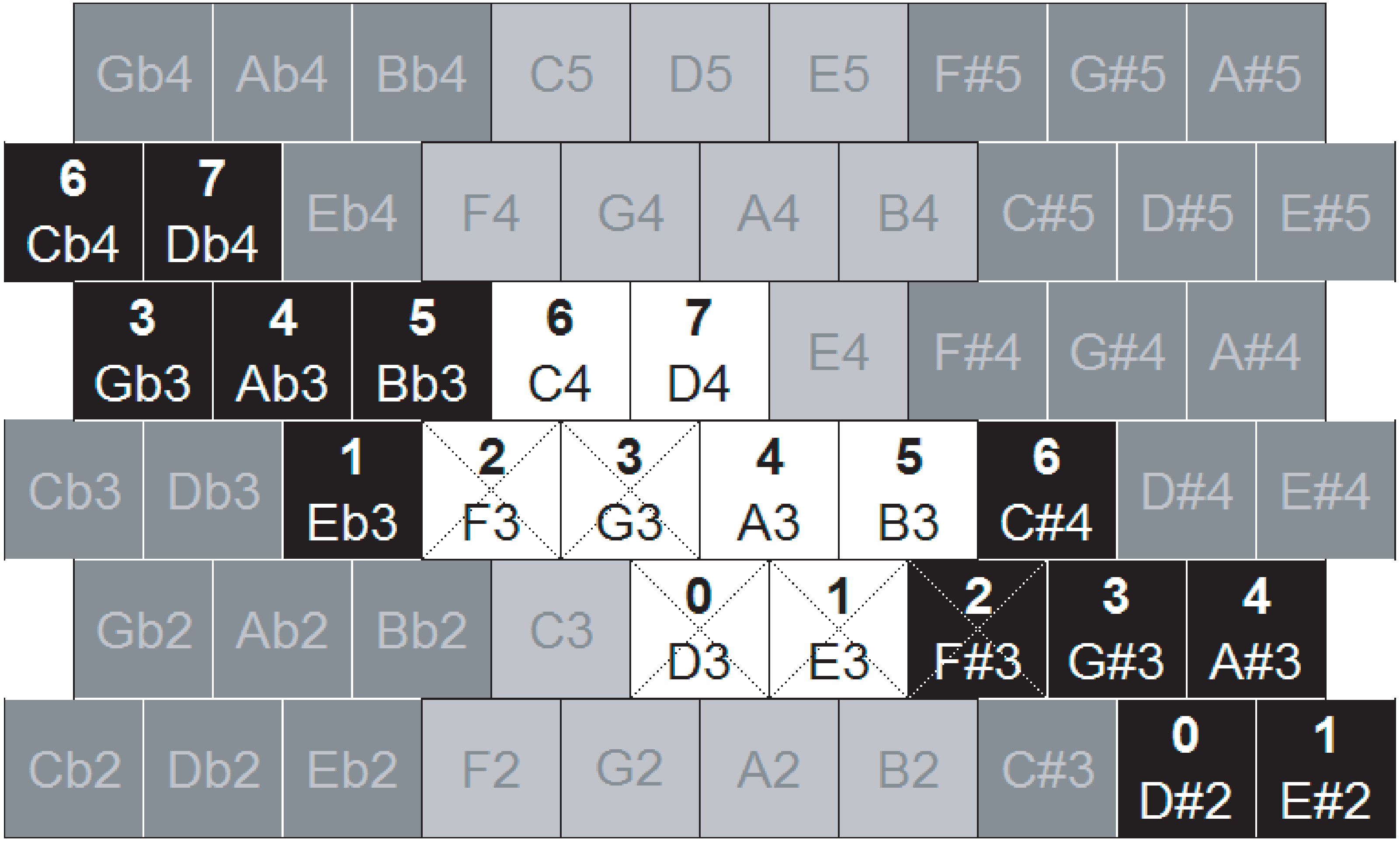
The boundaries of this tuning range mark the tuning points at which some of the scale steps shrink to unison, and therefore the tuning points at which the melodic identity of intervals in that scale changes.

Example: for the 12-note MOS scale where $\alpha = 2$ and $\beta \approx \frac{3}{2}$, (the chromatic scale that provides the background for common practice music) this range is $2^{\frac{4}{7}} < \beta < 2^{\frac{3}{5}}$, from 7-tet to 5-tet (same as well-formed scales).



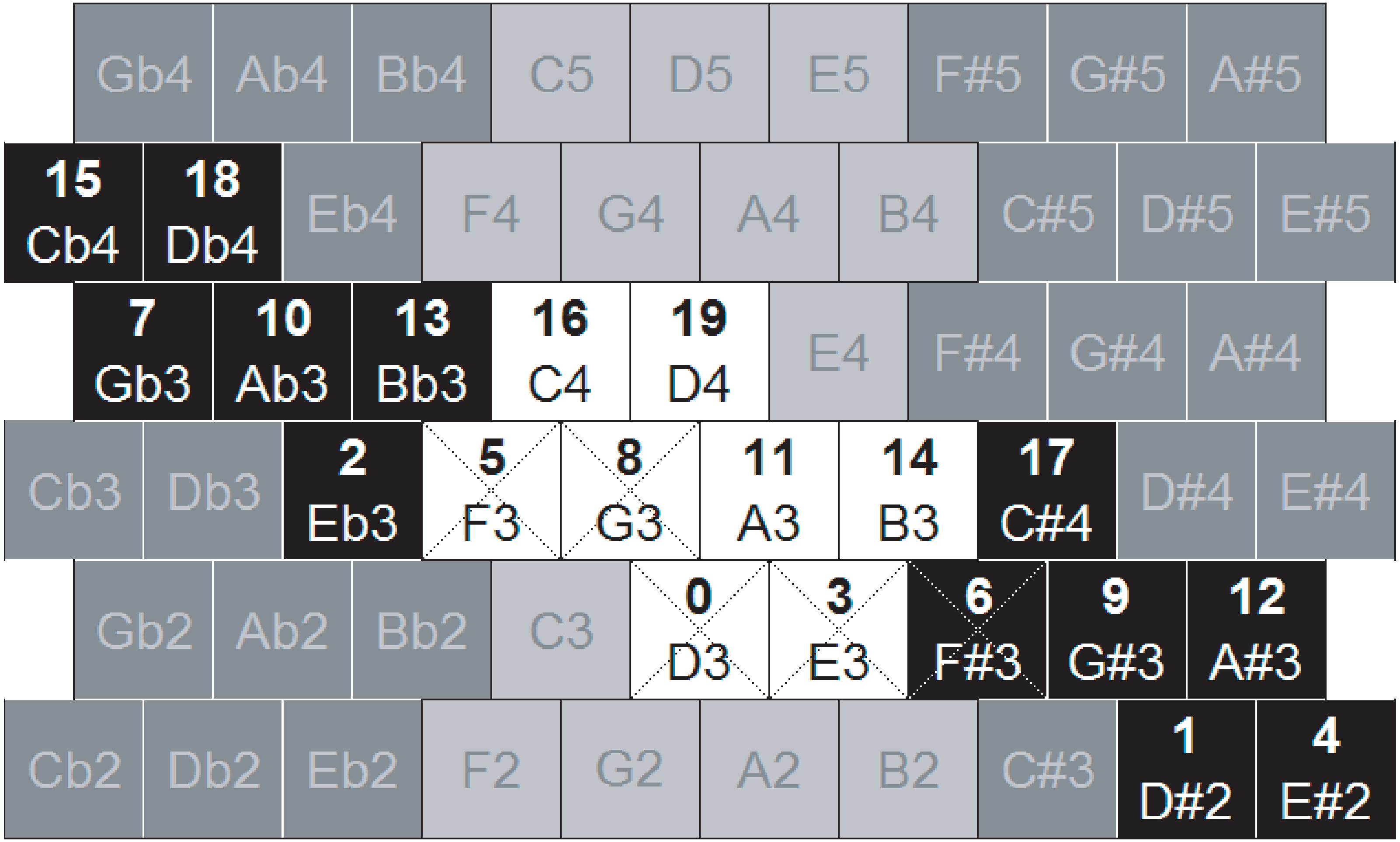
Valid Tuning Ranges: With $\alpha = 2$, the size of major second (M2), minor second (m2), and augmented unison (AU) over a range of β.

Through the Diatonic Tuning Continuum I



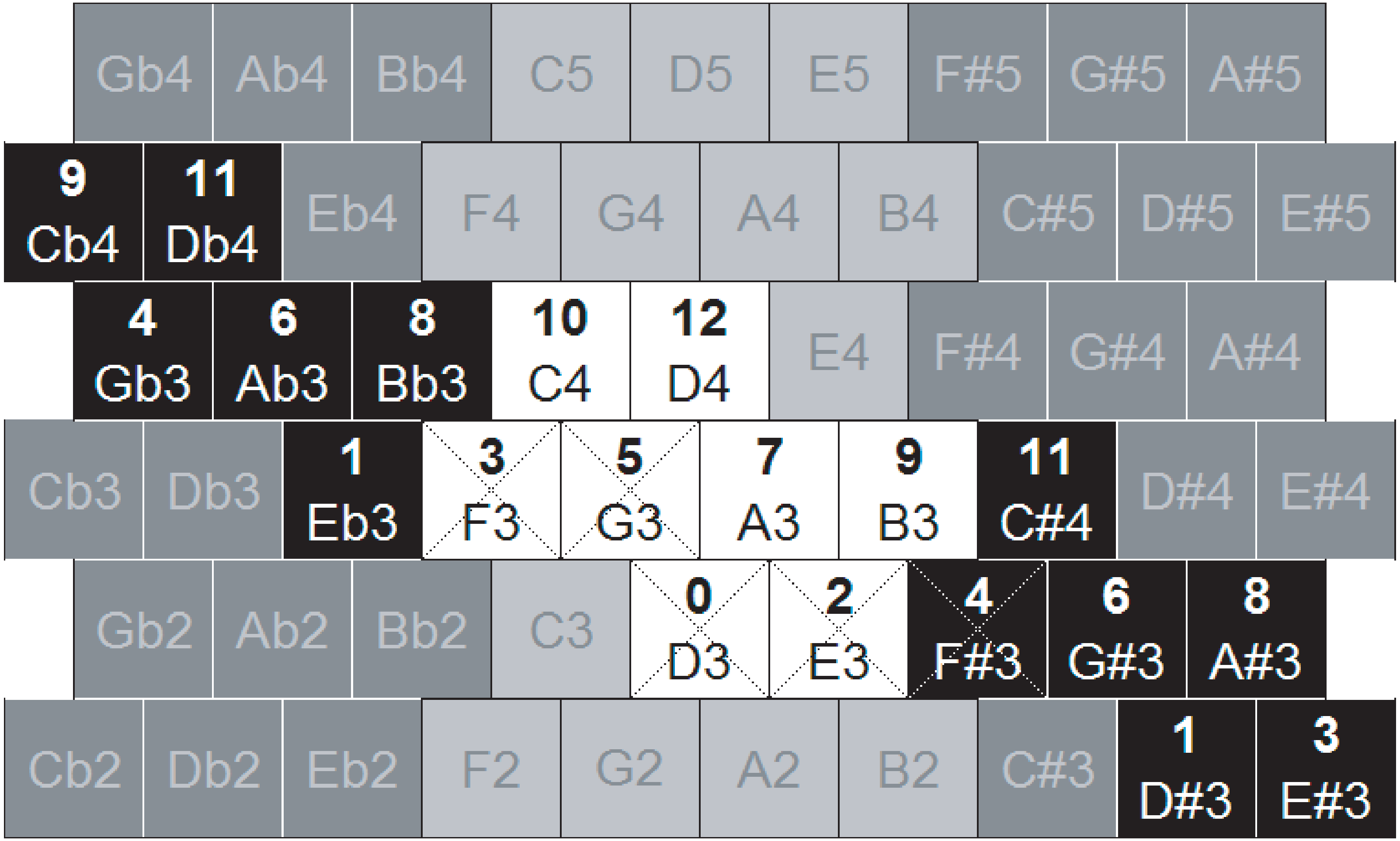
$\beta = 2^{\frac{4}{7}}$ (7-tet)

Through the Diatonic Tuning Continuum II



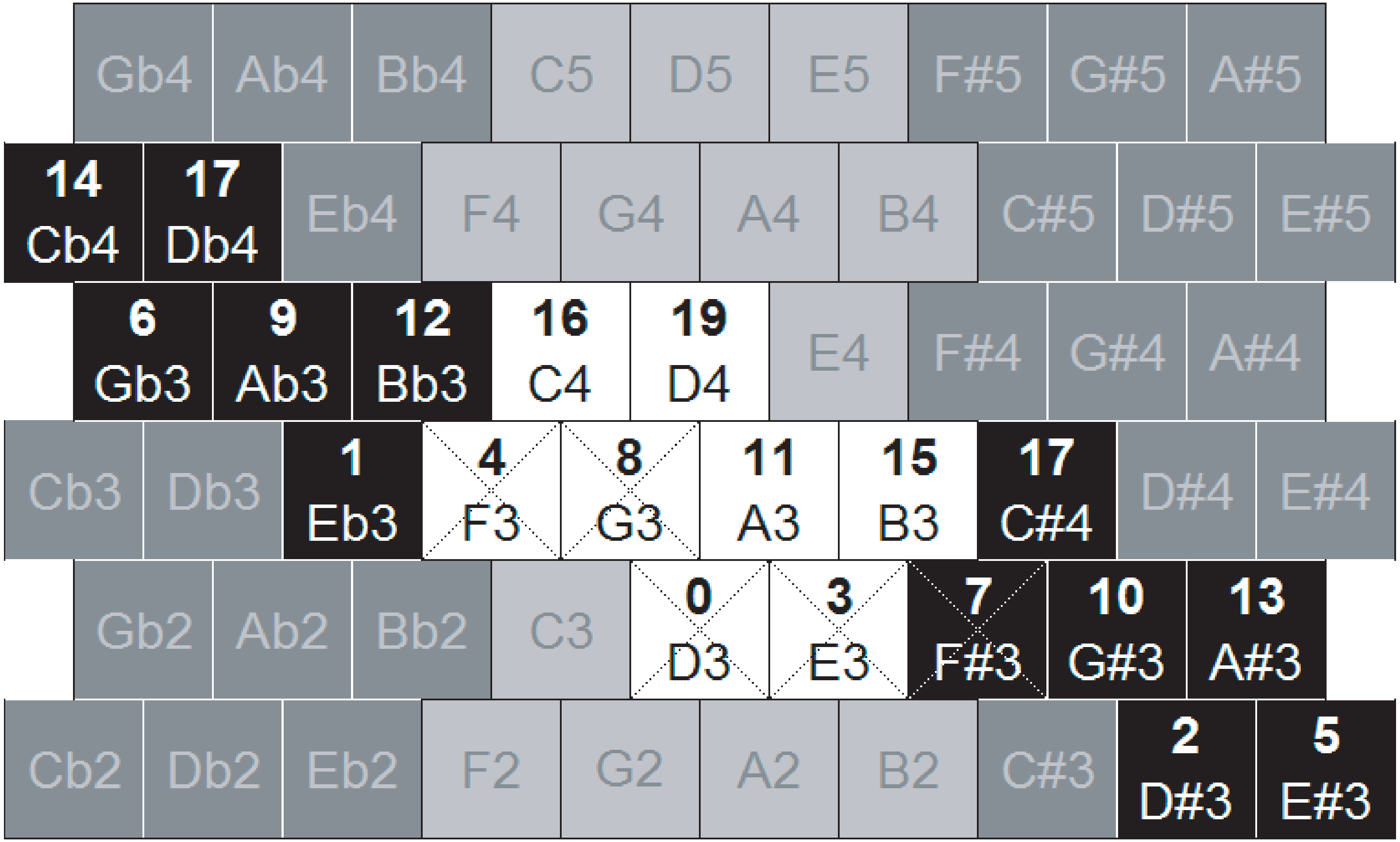
$$2^{\frac{4}{7}} < \beta < 2^{\frac{7}{12}}$$

Through the Diatonic Tuning Continuum III



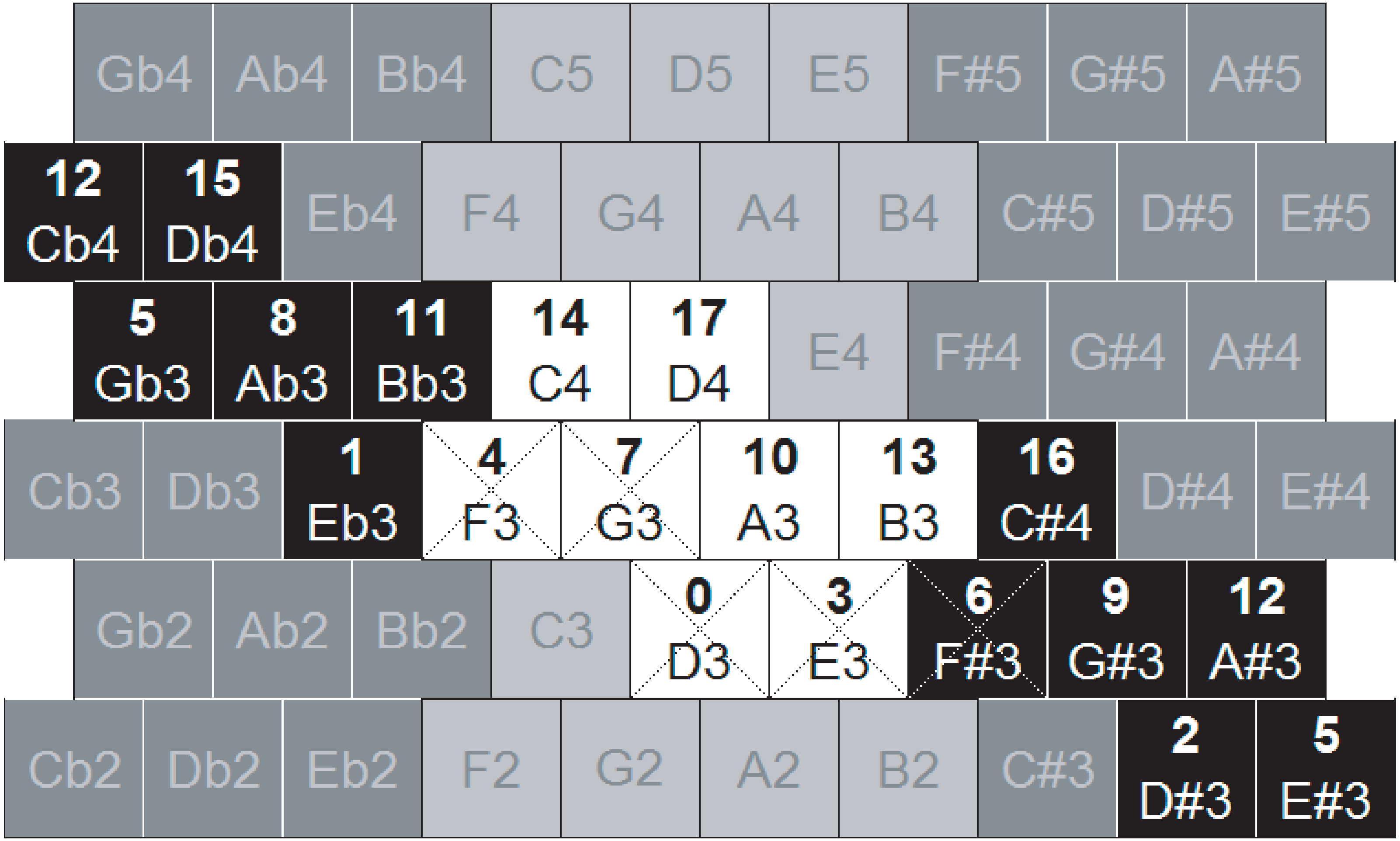
$\beta = 2^{\frac{7}{12}}$ (12-tet)

Through the Diatonic Tuning Continuum IV



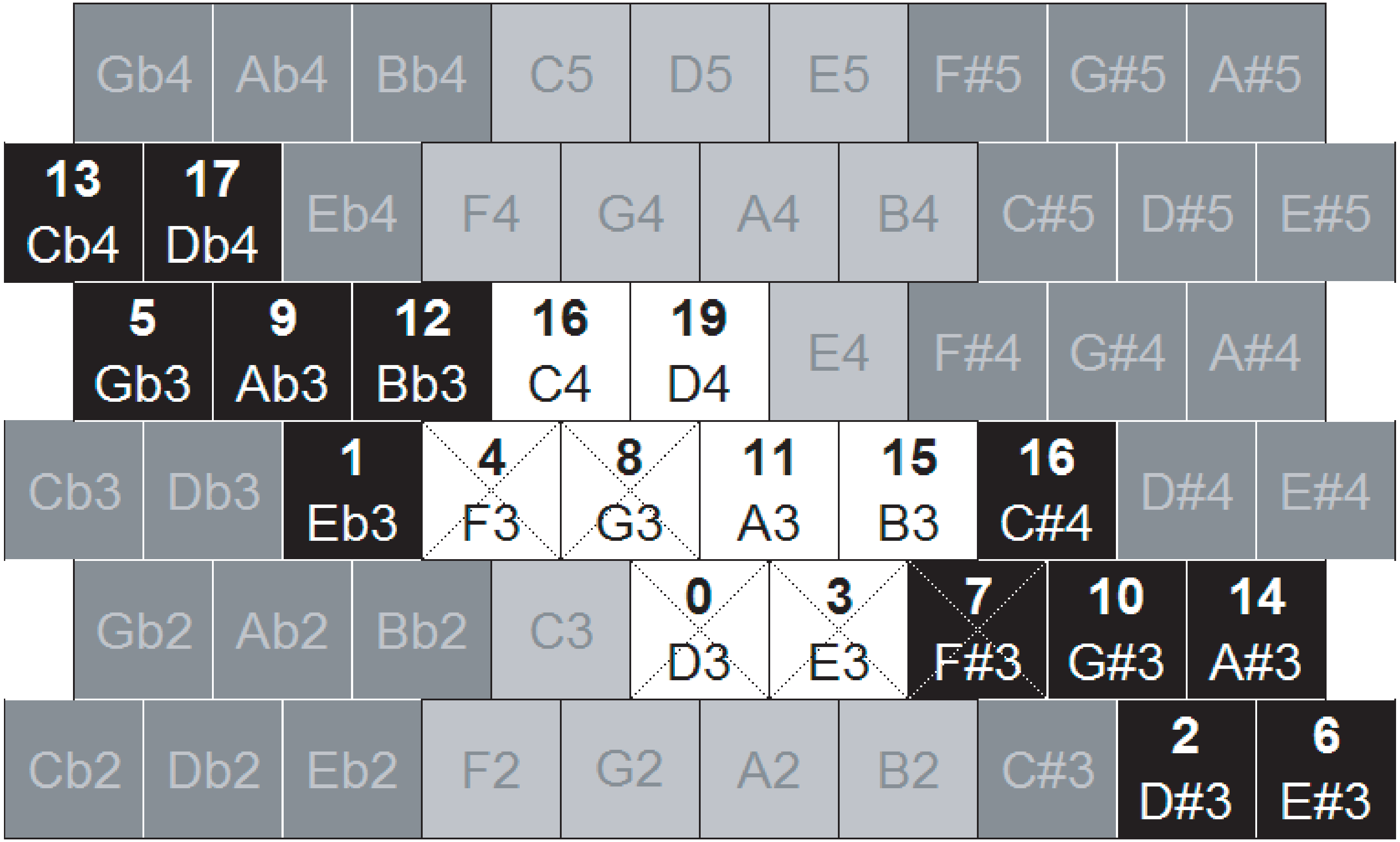
$$2^{\frac{7}{12}} < \beta < 2^{\frac{10}{17}}$$

Through the Diatonic Tuning Continuum V



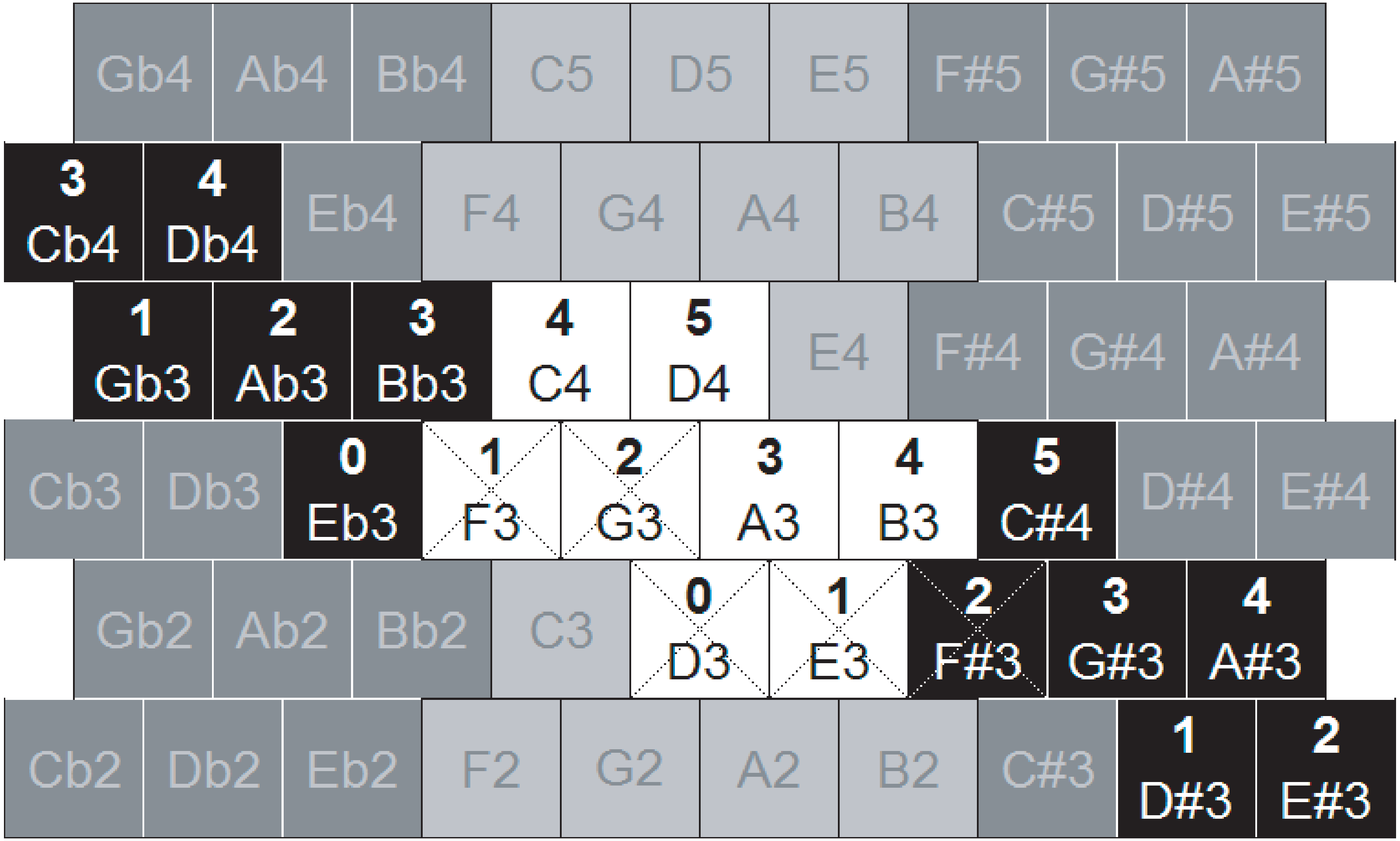
$\beta = 2^{\frac{10}{17}}$ (17-tet)

Through the Diatonic Tuning Continuum VI



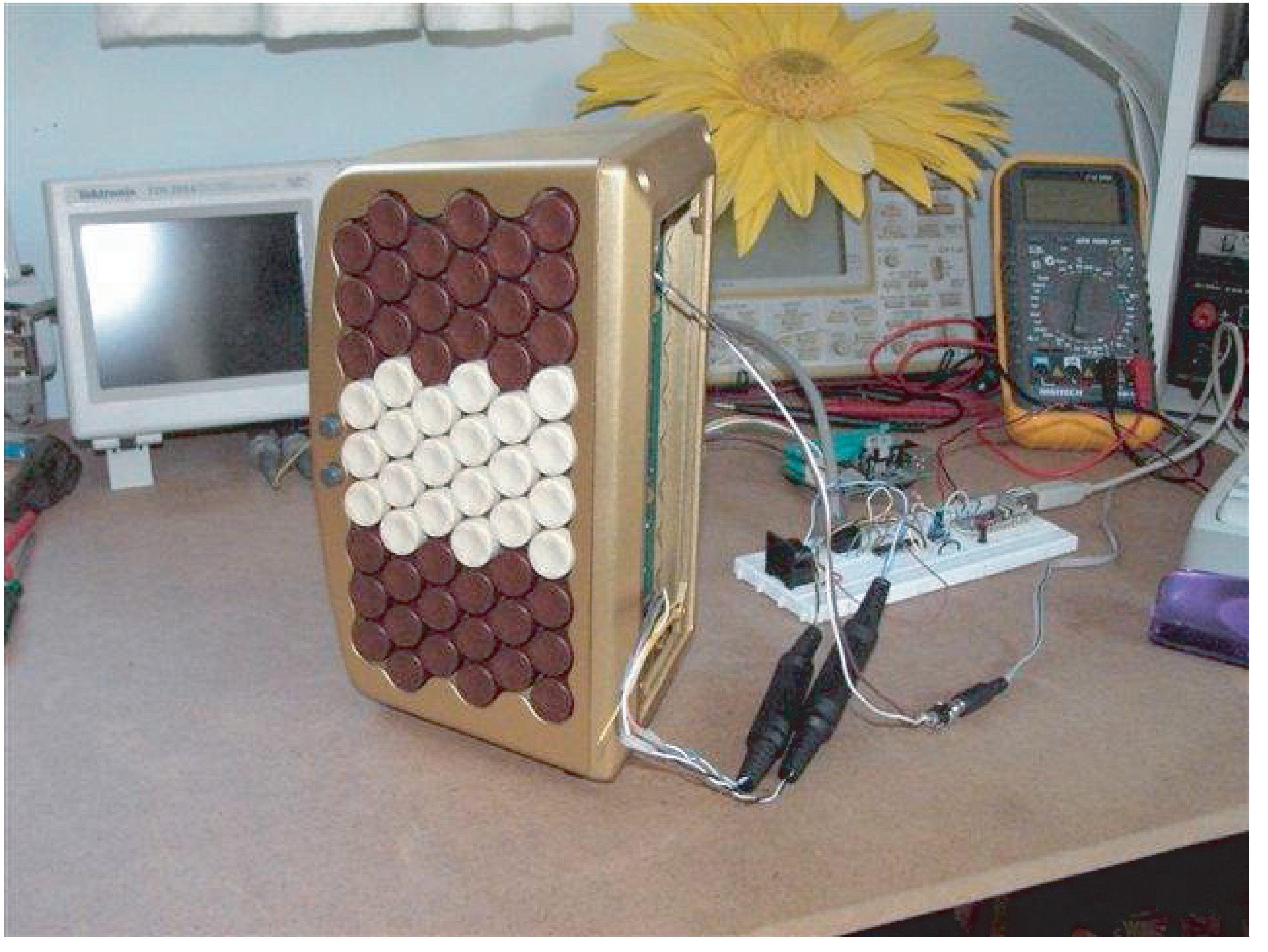
$$2^{\frac{10}{17}} < \beta < 2^{\frac{3}{5}}$$

Through the Diatonic Tuning Continuum VII



$\beta = 2^{\frac{3}{5}}$ (5-tet)

Prototype “Thummer” Keyboard



Summary

It is possible to design a keyboard capable of smoothly moving between various tunings, retaining the same fingerings in all tunings.

An example using the syntonic comma over the range of recognizable diatonic tunings illustrates, though other choices could be made.

The resulting keyboard layouts are transpositionally invariant as well as tuning invariant.

Along the way, we needed to understand the temperament mapping (based on commas), the layout mapping (linear), and to decide on a measure of the allowable range over which intervals are valid (we chose the “melodic” MOS method).