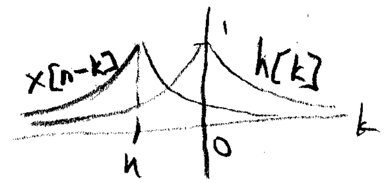
ECE 330
Spring 2008
Final Exam

Name: Solutions

Important Note: To receive full credit, you must show all your work and fully justify your answers.

1) Consider an LTI system with impulse response $h[n] = e^{-|n|}$ and input $x[n] = e^{-|n|}$. Find the output $y[n]$.

$$y[n] = e^{-|n|} * e^{-|n|}$$



Case I: $n<0$

$$y[n] = \sum_{k=-\infty}^{n} e^{k-n} e^{k} + \sum_{k=n+1}^{0} e^{n-k} e^{k} + \sum_{k=1}^{\infty} e^{n-k} e^{-k}$$

$$= \sum_{i=0}^{\infty} e^{n-2i} + e^{n} \sum_{k=n+1}^{0} 1 + \sum_{i=0}^{\infty} e^{n-2i-2}$$

$$= \frac{e^{n}}{1-e^{-2}} - n e^{n} + \frac{e^{n-2}}{1-e^{-2}}$$

$$= \left(\frac{e^2+1}{e^2-1} - n \right) e^{n}$$

Case II: $n \geq 0$

By symmetry,

$$y[n] = \left(\frac{e^2+1}{e^2-1} + n \right) e^{-n}$$

Overall,

$$y[n] = \left(\frac{e^2+1}{e^2-1} + |n| \right) e^{-|n|}$$

3) Find

a) the Laplace transform of $e^{-|t|}u\left(1-t^2\right)$ and its region of convergence,

$$X(s) = \int_{-1}^{1} e^{-|t|} e^{-st}\,dt = \int_{-1}^{0} e^{t} e^{-st}\,dt + \int_{0}^{1} e^{-t} e^{-st}\,dt$$

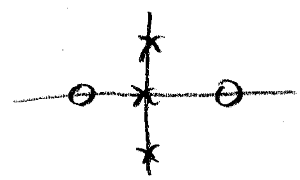
$$= \frac{1}{1-s} e^{(1-s)t}\Big|_{-1}^{0} - \frac{1}{1+s} e^{-(1+s)t}\Big|_{0}^{1}$$

$$= \frac{1-e^{s-1}}{1-s} - \frac{e^{-(s+1)}-1}{1+s}$$

ROC = entire plane

b) the inverse Laplace transform of $\frac{s^2-1}{s(s^2+1)}$ with region of convergence $\operatorname{Re} s < 0$,

$$\frac{s^2-1}{s(s^2+1)} = \frac{A_{11}}{s} + \frac{A_{21}}{s+j} + \frac{A_{21}^*}{s-j}$$



$$A_{11} = \frac{s^2-1}{s^2+1}\Big|_{s=0} = -1, \quad A_{21} = \frac{s^2-1}{s(s-j)}\Big|_{s=-j} = 1$$

$$x(t) = \left(1 - e^{-jt} - e^{jt}\right)u(-t) = (1-2\cos t)\,u(-t)$$

c) the Laplace transform of $\sum_{n=-\infty}^{\infty} e^{-|n|}\delta(t-n)$ and its region of convergence.

$$\delta(t-n) \leftrightarrow e^{-sn}$$

$$X(s) = \sum_{n=-\infty}^{\infty} e^{-|n|} e^{-sn}$$

$$= \sum_{n=-\infty}^{0} e^{n} e^{-sn} + \sum_{n=0}^{\infty} e^{-n} e^{-sn} - 1$$

$$= \sum_{n=0}^{\infty} e^{(s-1)n} + \sum_{n=0}^{\infty} e^{-(s+1)n} - 1$$

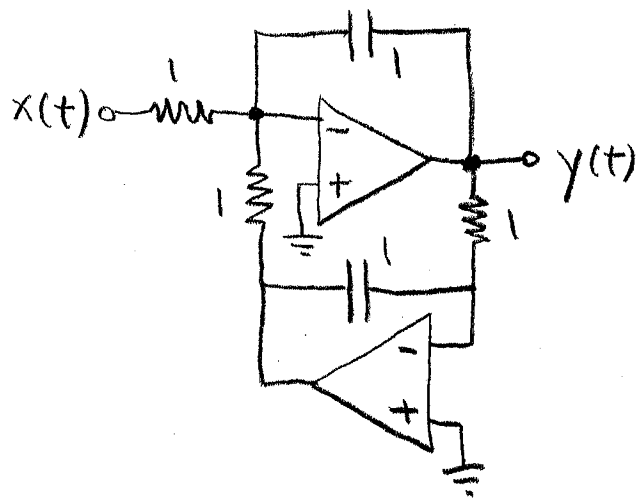
$$= \frac{1}{1-e^{s-1}} + \frac{1}{1-e^{-(s+1)}} - 1$$

The series converge for $|e^{s-1}|, |e^{-(s+1)}| < 1$.

Equivalently, $\mathrm{Re}(s-1), -\mathrm{Re}(s+1) < 0$.

$$-1 < \mathrm{Re}\, s < 1$$

4)



The op amp circuit shown is governed by the differential equation

$$\frac{d^2y}{dt^2} - y = -\frac{dx}{dt}.$$

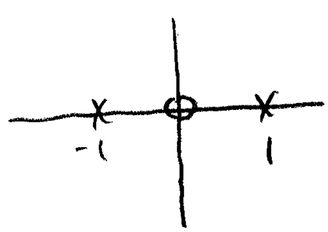
a) Find the transfer function $H(s)$ of the system.

$$H(s) = -\frac{s}{s^2-1}$$

b) Plot the poles, zeros, and region of convergence of H.

zeros: 0

poles: ± 1



The circuit is causal, so the ROC is

$$\mathrm{Re}\, s > 1$$

c) Find the unit step response.

$$s(t) = \mathcal{L}^{-1}\left\{-\frac{1}{s^2-1}\right\}$$

$$-\frac{1}{s^2-1} = \frac{A_{11}}{s+1} + \frac{A_{21}}{s-1}$$

$$A_{11} = -\frac{1}{s-1}\Big|_{s=-1} = \frac{1}{2}, \quad A_{21} = -\frac{1}{s+1}\Big|_{s=1} = -\frac{1}{2}$$

$$s(t) = \frac{1}{2}(e^{-t} - e^{t})u(t)$$

d) Find the impulse response.

$$h(t) = \mathcal{L}^{-1}\left\{-\frac{s}{s^2-1}\right\}$$

$$-\frac{s}{s^2-1} = \frac{A_{11}}{s+1} + \frac{A_{21}}{s-1}$$

$$A_{11} = -\frac{s}{s-1}\Big|_{s=-1} = -\frac{1}{2}, \quad A_{21} = -\frac{s}{s+1}\Big|_{s=1} = -\frac{1}{2}$$

$$h(t) = -\frac{1}{2}(e^{-t} + e^{t})u(t)$$

e) Is the circuit BIBO stable? Explain.

The system is causal and has a pole p with $\mathrm{Re}\, p \geq 0$, so the system is unstable.