Determine which of the following signals are periodic. For each periodic signal, find its fundamental period. Be sure to fully justify your answers.

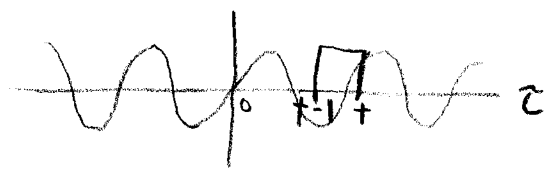
a) $x[n] = e^{j\frac{10}{7}\pi n}$

$$\frac{\Omega}{2\pi} = \frac{\frac{10}{7}\pi}{2\pi} = \frac{5}{7}$$

periodic w/ fundamental period $= 7$

b) $x(t) = (u(t)-u(t-1))*\sin(t)$

$$x(t) = \int_{t-1}^{t} \sin\tau \, d\tau$$



$$= \cos(t-1) - \cos(t)$$

periodic w/ fundamental period $= 2\pi$

c) $x[n] = e^{jn^2}$

For periodicity, there must exist an integer $N > 0$ such that $e^{j(n+N)^2} = e^{jn^2}$ for all n.

$$e^{j(n+N)^2} = e^{jn^2} \cdot e^{j(2n+N)N} = e^{jn^2}$$

$$e^{j(2n+N)N} = 1$$

$$(2n+N)N = 2\pi k \quad (k \text{ an integer})$$

$$\frac{(2n+N)N}{2k} = \pi$$

This is impossible, since π is irrational. $\Rightarrow$ not periodic

Determine whether each of the following signals is periodic. If it is periodic, find the fundamental period.

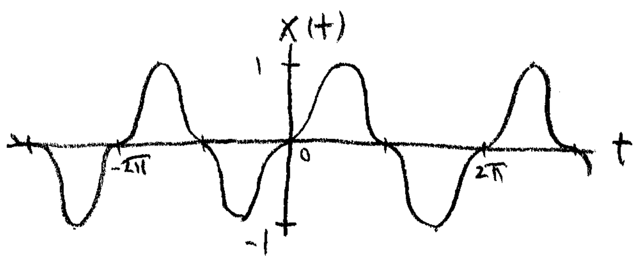
a) $e^{j\frac{10}{9}\pi n}$

$$\frac{\frac{10}{9}\pi}{2\pi} = \frac{5}{9} \text{ is rational} \Rightarrow \text{periodic}$$

$$N_0 = 9$$

b) $\dfrac{\sin(t)}{1+\cos^2(t)}$

sin t and cos t have common period 2π, so $x(t)$ is periodic.



$$T_0 = 2\pi$$

c) $e^{jn^{100}}$

If $x[n]$ has period N,

$$e^{j(n+N)^{100}} = e^{jn^{100}}$$

$$e^{j((n+N)^{100} - n^{100})} = 1$$

$$(n+N)^{100} - n^{100} = 2\pi k$$

$$\frac{(n+N)^{100} - n^{100}}{2k} = \pi$$

This leads to a contradiction, so $x[n]$ is not periodic.

rational

irrational

Consider the system governed by

$$y[n] = \sum_{k=0}^{\infty} 2^{-k} x^2[n-k].$$

Determine whether the system is

a) linear

$$\alpha x[n] \longrightarrow \sum_{k=0}^{\infty} 2^{-k} (\alpha x[n-k])^2$$

$$= \alpha^2 \sum_{k=0}^{\infty} 2^{-k} x^2[n-k]$$

$$= \alpha^2 y[n]$$

$$\neq \alpha y[n]$$

nonlinear

b) time-invariant

$$x_1[n] = x[n+N]$$

$$y_1[n] = \sum_{k=0}^{\infty} 2^{-k} x_1^2[n-k]$$

$$= \sum_{k=0}^{\infty} 2^{-k} x^2[n-k+N]$$

$$y[n+N] = \sum_{k=0}^{\infty} 2^{-k} x^2[n+N-k]$$

$$= y_1[n]$$

time-invariant

c) causal

For each n, the summation involves $x[n]$, $x[n-1]$, $x[n-2]$, ... (only past and present inputs).

causal

d) dynamic

From c), the summation involves past inputs.

dynamic

e) BIBO stable

Suppose $|x[n]| < M$. Then

$$|y[n]| = \left| \sum_{k=0}^{\infty} 2^{-k} x^2[n-k] \right| \leq \sum_{k=0}^{\infty} 2^{-k} |x^2[n-k]|$$

$$= \sum_{k=0}^{\infty} 2^{-k} |x[n-k]|^2 < M^2 \sum_{k=0}^{\infty} 2^{-k}$$

$$= M^2 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2M^2$$

stable

Two systems are connected in series as shown.

$$x[n] \longrightarrow \boxed{\text{cube}} \xrightarrow{x^3[n]} \boxed{\text{accumulate}} \longrightarrow y[n]$$

The output is given by

$$y[n] = \sum_{k=-\infty}^{n} x^3[k].$$

Determine whether the overall system satisfies each of the following system properties. Be sure to fully justify your answers.

a) Linearity

Nonlinear, since

$$\alpha x[n] \longrightarrow \sum_{k=-\infty}^{n} \alpha^3 x^3[k] = \alpha^3 y[n] \neq \alpha y[n]$$

b) Causality

Causal, since $y[n]$ depends only on $x[k]$ for $k \leq n$.

c) BIBO Stability

Let $x[n] = u[n]$. (bounded)

$y[n] = \sum_{k=-\infty}^{n} u^3[k] = \sum_{k=-\infty}^{n} u[k]$

For $n \geq 0$, $y[n] = \sum_{k=0}^{n} 1 = n+1$ (unbounded)

The system is unstable.

d) Time-invariance

Let $x_1[n] = x[n+N]$. Then

$y_1[n] = \sum_{k=-\infty}^{n} x_1^3[k] = \sum_{k=-\infty}^{n} x^3[k+N]$

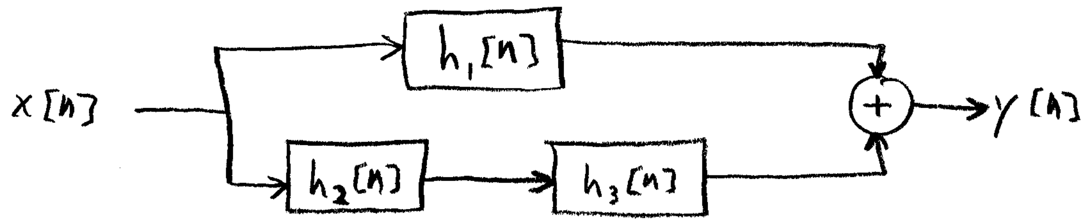
Let $i = k+N$.

$y_1[n] = \sum_{i=-\infty}^{n+N} x^3[i] = y[n+N] \Rightarrow$ Time-invariant

e) Dynamic

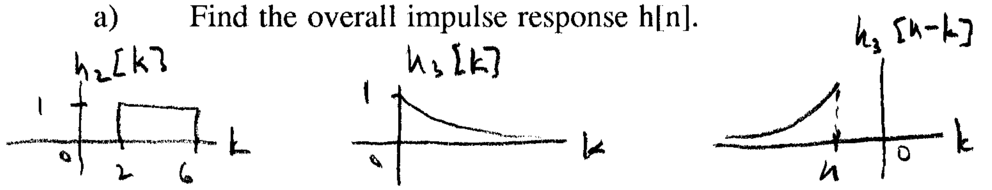
Dynamic, since $y[n]$ depends on $x[n-1], x[n-2], \ldots$

Consider the interconnected LTI system shown:



Let $h_1[n]=u[n]$, $h_2[n]=u[n-2]-u[n-7]$, and $h_3[n]=2^{-n}u[n]$.

a) Find the overall impulse response h[n].



Case I: $n < 2$ $\quad h_2 * h_3 = 0$

Case II: $2 \le n \le 6$ $\quad h_2 * h_3 = \sum_{k=2}^{n} 2^{-(n-k)} = 2^{2-n}\sum_{k=0}^{n-2} 2^{k} = 2^{2-n}\left(\frac{1-2^{n-1}}{1-2}\right) = 2(1-2^{1-n})$

Case III: $n > 6$ $\quad h_2 * h_3 = \sum_{k=2}^{6} 2^{-(n-k)} = 2^{2-n}\sum_{k=0}^{4} 2^{k} = 2^{2-n}\left(\frac{1-2^{5}}{1-2}\right) = 124 \cdot 2^{-n}$

$$h[n] = h_1[n] + h_2[n] * h_3[n] = \begin{cases} 0, & n < 0 \\ 1, & n = 1 \\ 1 + 2(1-2^{1-n}), & 2 \le n \le 6 \\ 1 + 124 \cdot 2^{-n}, & n > 6 \end{cases}$$

b) Is the system causal or noncausal?

Causal, since $h[n] = 0$ for $n < 0$.

c) Is the system dynamic or static?

Dynamic, since $h[n] \neq 0$ for $n > 0$.

d) Is the system stable or unstable?

Unstable, since $h[n] \not\to 0$ as $n \to \infty$.

$\left(\sum_{n=-\infty}^{\infty} |h[n]| = \infty \right)$

An LTI DT system has impulse response

$$h[n] = 2^{-n} u[n+1].$$

Find the output y[n] of the system resulting from the input

$$x[n] = 3^{-n} u[n-1].$$

$$x[k]h[n-k] = 3^{-k} u[k-1]\, 2^{-(n-k)} u[n-k+1]$$

$$= \left(\tfrac{1}{2}\right)^n \left(\tfrac{2}{3}\right)^k u[k-1]\, u[n-k+1]$$

$$= \begin{cases} \left(\tfrac{1}{2}\right)^n \left(\tfrac{2}{3}\right)^k, & 1 \le k \le n+1 \\ 0, & \text{otherwise} \end{cases}$$

$$n \ge 0 \Rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=1}^{n+1} \left(\tfrac{1}{2}\right)^n \left(\tfrac{2}{3}\right)^k$$

$$= \left(\tfrac{1}{2}\right)^n \sum_{i=0}^{n} \left(\tfrac{2}{3}\right)^{i+1} \qquad (i = k-1)$$

$$= \tfrac{2}{3}\left(\tfrac{1}{2}\right)^n \sum_{i=0}^{n} \left(\tfrac{2}{3}\right)^{i}$$

$$= \tfrac{2}{3}\left(\tfrac{1}{2}\right)^n \frac{1-\left(\tfrac{2}{3}\right)^{n+1}}{1-\tfrac{2}{3}}$$

$$= 2\left(\tfrac{1}{2}\right)^n \left(1-\left(\tfrac{2}{3}\right)^{n+1}\right)$$

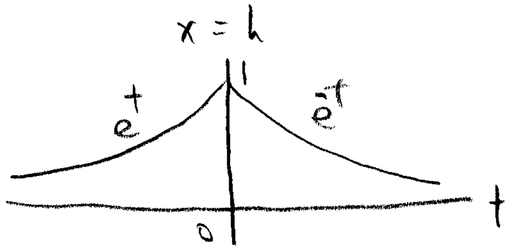
$$n \text{ arbitrary} \Rightarrow \quad y[n] = 2\left(\tfrac{1}{2}\right)^n \left(1-\left(\tfrac{2}{3}\right)^{n+1}\right) u[n]$$

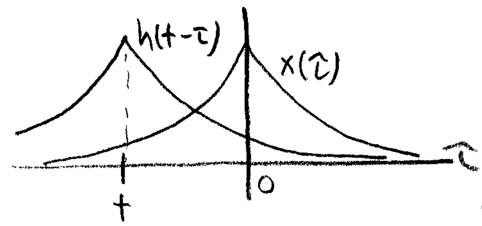
Using time-domain methods exclusively, find the convolution y(t) of the functions

$$x(t) = e^{-|t|}$$

$$h(t) = e^{-|t|}.$$

Sketch y(t). (Hint: $\int te^{-t}dt = -e^{-t}(t+1)$, $\int te^{t}dt = e^{t}(t-1)$)





Case I: $t<0$

$$x * h = \int_{-\infty}^{t} e^{-(t-\tau)} e^{\tau} d\tau + \int_{t}^{0} e^{t-\tau} e^{\tau} d\tau + \int_{0}^{\infty} e^{t-\tau} e^{-\tau} d\tau$$

$$= e^{-t}\int_{-\infty}^{t} e^{2\tau} d\tau + e^{t}\int_{t}^{0} 1\, d\tau + e^{t}\int_{0}^{\infty} e^{-2\tau} d\tau$$

$$= \tfrac{1}{2}e^{t} - t e^{t} + \tfrac{1}{2}e^{t}$$

$$= (1-t)e^{t}$$

Case II: $t>0$

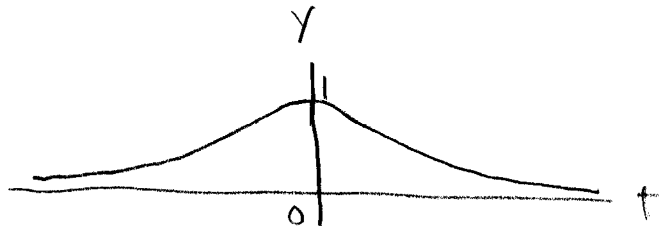
$$x * h = \int_{-\infty}^{0} e^{-(t-\tau)} e^{\tau} d\tau + \int_{0}^{t} e^{-(t-\tau)} e^{-\tau} d\tau + \int_{t}^{\infty} e^{t-\tau} e^{-\tau} d\tau$$

$$= e^{-t}\int_{-\infty}^{0} e^{2\tau} d\tau + e^{-t}\int_{0}^{t} 1\, d\tau + e^{t}\int_{t}^{\infty} e^{-2\tau} d\tau$$

$$= \tfrac{1}{2}e^{-t} + t e^{-t} + \tfrac{1}{2}e^{-t}$$

$$= (1+t)e^{-t}$$

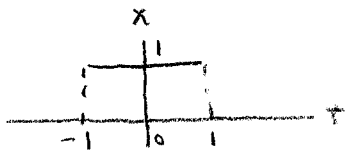
$$y(t) = (1+|t|)e^{-|t|}$$

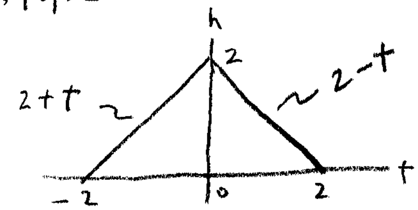


Using time-domain techniques exclusively, find the convolution of the following signals:

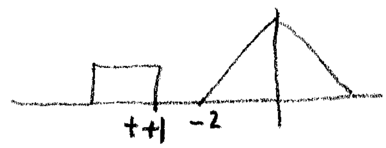
$$x(t) = \begin{cases} 1, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

$$h(t) = \begin{cases} 2-|t|, & |t| \le 2 \\ 0, & |t| > 2 \end{cases}$$





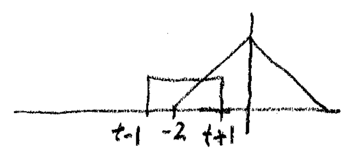
Case I: $t < -3$, $y(t) = 0$



Case II: $-3 \le t < -1$

$$y(t) = \int_{-2}^{t+1} 2+\tau \, d\tau$$

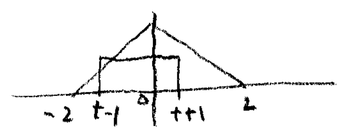
$$= \left(2\tau + \frac{\tau^2}{2}\right)\Big|_{-2}^{t+1} = \frac{t^2}{2} + 3t + \frac{9}{2}$$



Case III: $-1 \le t < 1$

$$y(t) = \int_{t-1}^{0} 2+\tau \, d\tau + \int_{0}^{t+1} 2-\tau \, d\tau$$

$$= \left(2\tau + \frac{\tau^2}{2}\right)\Big|_{t-1}^{0} + \left(2\tau - \frac{\tau^2}{2}\right)\Big|_{0}^{t+1} = 3 - t^2$$

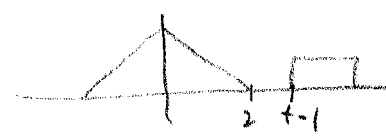


Case IV: $1 \le t < 3$

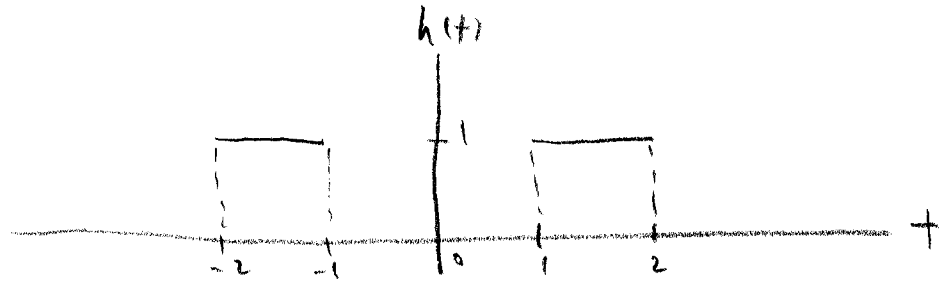
$$y(t) = \int_{t-1}^{2} 2-\tau \, d\tau$$

$$= \left(2\tau - \frac{\tau^2}{2}\right)\Big|_{t-1}^{2} = \frac{t^2}{2} - 3t + \frac{9}{2}$$

Case V: $3 \le t$, $y(t) = 0$

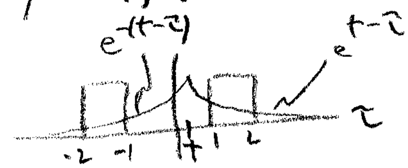


Consider the LTI CT system with impulse response h(t) depicted below.



Using time-domain techniques exclusively, find the output y(t) of the system resulting from the input $x(t) = e^{-|t|}$.

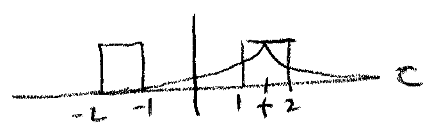
$y(t) = h(t) * x(t)$ is even, so we need only consider $t \geq 0$. For $t < 0$, $y(t) = y(|t|)$.



Case I: $0 \leq t < 1$

$$y(t) = \int_{-2}^{-1} e^{-(t-\tau)}\,d\tau + \int_{1}^{2} e^{t-\tau}\,d\tau$$

$$= e^{-t}(e^{-1} - e^{-2}) - e^{t}(e^{-2} - e^{-1})$$

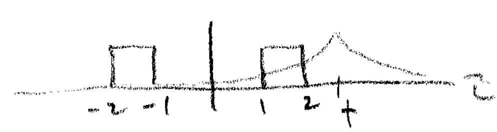


Case II: $1 \leq t < 2$

$$y(t) = \int_{-2}^{-1} e^{-(t-\tau)}\,d\tau + \int_{1}^{t} e^{-(t-\tau)}\,d\tau + \int_{t}^{2} e^{t-\tau}\,d\tau$$

$$= e^{-t}(e^{-1} - e^{-2}) + e^{-t}(e^{t} - e) - e^{t}(e^{-2} - e^{-t})$$

$$= 2 + e^{-t}(e^{-1} - e^{-2} - e) - e^{t-2}$$



Case III: $t \geq 2$

$$y(t) = \int_{-2}^{-1} e^{-(t-\tau)}\,d\tau + \int_{1}^{2} e^{-(t-\tau)}\,d\tau$$

$$= e^{-t}(e^{-1} - e^{-2}) + e^{-t}(e^{2} - e)$$

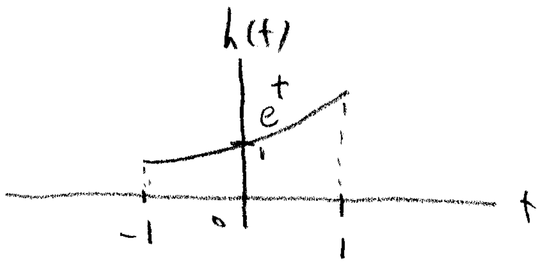
$$y(t) = \begin{cases} e^{-|t|}(e^{-1} - e^{-2}) - e^{|t|}(e^{-2} - e^{-1}), & 0 \leq |t| < 1 \\ 2 + e^{-|t|}(e^{-1} - e^{-2} - e) - e^{|t|-2}, & 1 \leq |t| < 2 \\ e^{-|t|}(e^{-1} - e^{-2} + e^{2} - e), & |t| \geq 2 \end{cases}$$

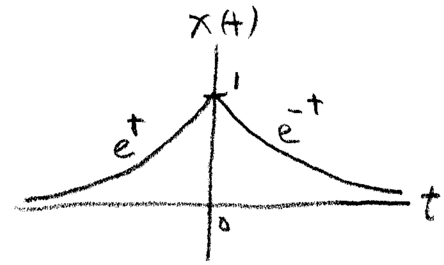
Consider the LTI system with impulse response

$$h(t) = e^t(u(t+1)-u(t-1)).$$

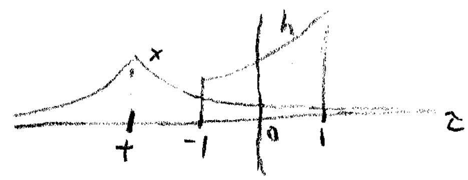
Find the output y(t) corresponding to the input

$$x(t) = e^{-|t|}.$$



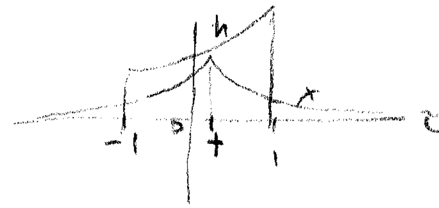


Case I: $t < -1$



$$y(t) = \int_{-1}^{1} e^{\tau} e^{t-\tau}\, d\tau = 2e^{t}$$

Case II: $-1 \le t < 1$

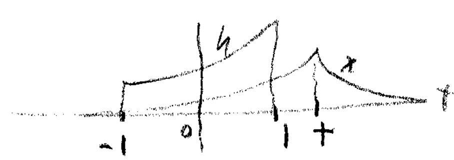


$$y(t) = \int_{-1}^{t} e^{\tau} e^{-(t-\tau)}\, d\tau + \int_{t}^{1} e^{\tau} e^{t-\tau}\, d\tau$$

$$= e^{-t}\int_{-1}^{t} e^{2\tau}\, d\tau + e^{t}\int_{t}^{1} 1\, d\tau = \frac{1}{2} e^{-t}(e^{2t} - e^{-2}) + e^{t}(1-t)$$

$$= \left(\frac{3}{2} - t\right)e^{t} - \frac{1}{2} e^{-(t+2)}$$

Case III: $t \ge 1$



$$y(t) = \int_{-1}^{1} e^{\tau} e^{-(t-\tau)}\, d\tau = e^{-t}\int_{-1}^{1} e^{2\tau}\, d\tau$$

$$= \frac{e^{2} - e^{-2}}{2} e^{-t}$$