2) Consider the system

$$y[n] = \sum_{k=-\infty}^{\infty} e^{-|n-k|} x[k].$$

Answer each of the following questions. Be sure to fully justify your answers.

$$y[n] = h[n] * x[n], \quad h[n] = e^{-|n|}$$

a) Is the system dynamic (i.e. does it have memory)?

$h[n] \neq 0$ for $n \neq 0$

Yes

b) Is the system causal?

$h[n] \neq 0$ for $n < 0$

No

c) Is the system BIBO stable?

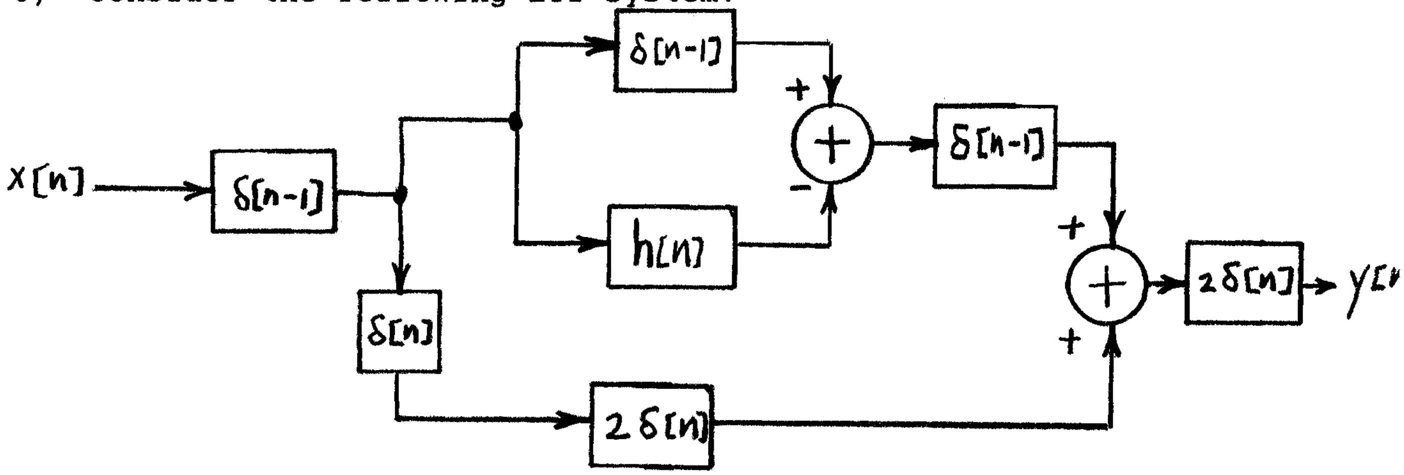
$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{0} e^{n} + \sum_{n=0}^{\infty} e^{-n} - 1 = 2\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^{n} - 1$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{1}{e}\right)^{n} - 1$$

$$\frac{1}{e} < 1 \Rightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Yes

6) Consider the following LTI system:



a) Express the output signal y[n] in terms of x[n] and h[n].

$$y[n] = 2\delta[n] * \Big(\delta[n-1] * \big(\delta[n-1] + h[n]\big) + 2\delta[n] * \delta[n] \Big) * \delta[n-1] * x[n]$$

$$= 2\big(\delta[n-2] + h[n-1] + 2\delta[n] \big) * x[n-1]$$

$$= 2x[n-3] + 4x[n-1] + 2h[n-1] * x[n-1]$$

$$= 2x[n-3] + 4x[n-1] + 2h[n] * x[n-2]$$

Consider the causal LTI described by the differential equation

$$y^{(4)} + \ddot{y} = \dot{x} + x.$$

b) Find the impulse response h(t) of the system.

$$s^4 + s^2 = s^2(s+j)(s-j)$$

$$h_1(t) = \alpha_{10} + \alpha_{11} t + \alpha_{20} e^{-jt} + \alpha_{30} e^{jt}$$

$$\left.\begin{aligned} h_1(0) &= \alpha_{10} + \alpha_{20} + \alpha_{30} = 0 \\ \dot{h}_1(0) &= \alpha_{11} - j\alpha_{20} + j\alpha_{30} = 0 \\ \ddot{h}_1(0) &= -\alpha_{20} - \alpha_{30} = 0 \\ \dddot{h}_1(0) &= j\alpha_{20} - j\alpha_{30} = 1 \end{aligned}\right\} \qquad \begin{aligned} \alpha_{30} &= \tfrac{j}{2} \\ \alpha_{20} &= -\tfrac{j}{2} \\ \alpha_{10} &= 0 \\ \alpha_{11} &= 1 \end{aligned}$$

$$h_1(t) = \left(t - \tfrac{j}{2} e^{-jt} + \tfrac{j}{2} e^{jt}\right) u(t) = (t - \sin t)\, u(t)$$

$$h(t) = \dot{h}_1(t) + h_1(t) = (1 - \cos t)\, u(t) + (t - \sin t)\,\delta(t) + (t - \sin t)\, u(t)$$

$$= (1 + t - \sin t - \cos t)\, u(t)$$

2) Consider the "full-wave rectified sine wave" $x(t)=|\sin(t)|$.

a) Find the Fourier coefficients of x(t).

$$T=\pi, \quad \omega_0 = \frac{2\pi}{T} = 2$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} \sin t \, e^{-jk2t}\,dt = \frac{1}{j2\pi}\int_0^{\pi}\left(e^{j(1-2k)t} - e^{-j(1+2k)t}\right)dt$$

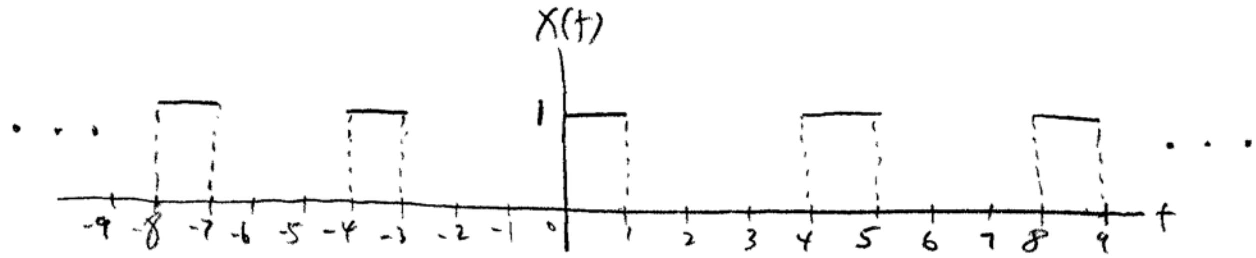
$$= \frac{1}{j2\pi}\left(\frac{1}{j(1-2k)}\left(e^{j(1-2k)\pi}-1\right) + \frac{1}{j(1+2k)}\left(e^{-j(1+2k)\pi}-1\right)\right)$$

$$= \frac{1}{j2\pi}\left(-\frac{2}{j(1-2k)} - \frac{2}{j(1+2k)}\right) = \frac{2}{\pi(1-4k^2)}$$

b) Find the Fourier series of x(t).

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \frac{2}{\pi}\sum_{k=-\infty}^{\infty} \frac{1}{1-4k^2} e^{j2kt}$$

Consider the periodic signal x(t) depicted below:



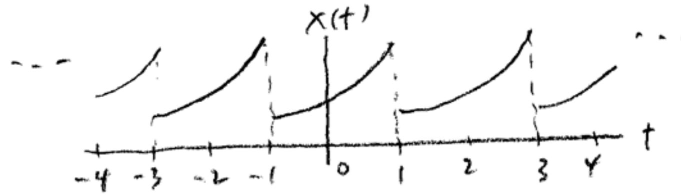
a) Find the Fourier coefficients a_k of x(t).

$$T_0 = 4, \quad \omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{2}$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{4}\int_0^{1} e^{-jk\frac{\pi}{2}t}\, dt$$

$$= \frac{j}{2\pi k}\left(e^{-jk\frac{\pi}{2}} - 1\right)$$

a) Find the Fourier series representation of the CT signal x(t) with period T=2 and satisfying $x(t) = e^t$ for $|t| < 1$.



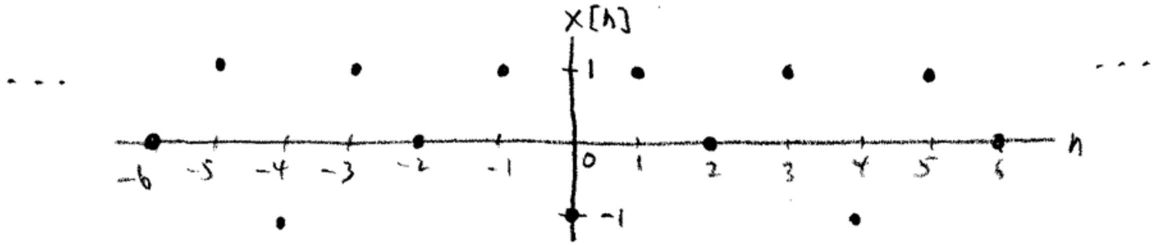
$\omega_0 = \pi$

$$a_k = \frac{1}{2}\int_{-1}^{1} e^{t} e^{-jk\pi t}\,dt$$

$$= \frac{1}{2(1-jk\pi)}\left(e(e^{-j\pi})^k - \frac{1}{e}(e^{j\pi})^k\right) = \frac{(e-\frac{1}{e})(-1)^k}{2(1-jk\pi)}$$

$$x(t) = \frac{e-\frac{1}{e}}{2}\sum_{k=-\infty}^{\infty}\frac{(-1)^k}{1-jk\pi}e^{jk\pi t}$$

1a) Find the Fourier series of the DT signal x[n] shown.



$N = 4, \omega_0 = \frac{\pi}{2}$

$$a_k = \frac{1}{4}\sum_{k=-1}^{2} x[n] e^{-jk\frac{\pi}{2}n}$$

$$= \frac{1}{4}\left(e^{j\frac{\pi}{2}k} - 1 + e^{-j\frac{\pi}{2}k}\right)$$

$$= \frac{1}{4}\left(2\cos\frac{\pi}{2}k - 1\right)$$

$$x[n] = \frac{1}{4}\sum_{n=0}^{3}\left(2\cos\frac{\pi}{2}k - 1\right)e^{jk\frac{\pi}{2}n}$$

b) Find the DT signal x[n] with period N=3 and Fourier coefficients

$$a_k = -2j\sin\left(\frac{2\pi}{3}k\right).$$

$\omega_0 = \frac{2\pi}{3}$

$$x[n] = \sum_{k=-1}^{1}\left(-2j\sin\left(\frac{2\pi}{3}k\right)\right)e^{jk\left(\frac{2\pi}{3}\right)n}$$

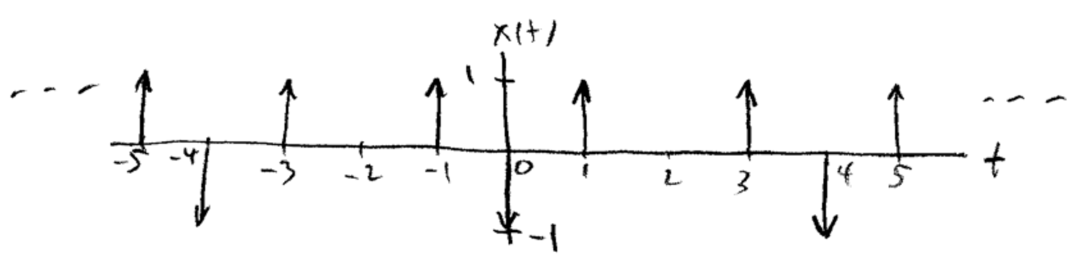
$$= -2j\sin\left(-\frac{2\pi}{3}\right)e^{-j\frac{2\pi}{3}n} - 2j\sin\left(\frac{2\pi}{3}\right)e^{j\frac{2\pi}{3}n}$$

$$= 2j\sin\frac{2\pi}{3}\left(e^{-j\frac{2\pi}{3}n} - e^{j\frac{2\pi}{3}n}\right)$$

$$= 2j\left(\frac{\sqrt{3}}{2}\right)\left(-2j\sin\frac{2\pi}{3}n\right) = 2\sqrt{3}\sin\frac{2\pi}{3}n$$

c) Find the CT signal x(t) whose fundamental period and Fourier coefficients are the same as those you found in part 1a).

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-n)$$



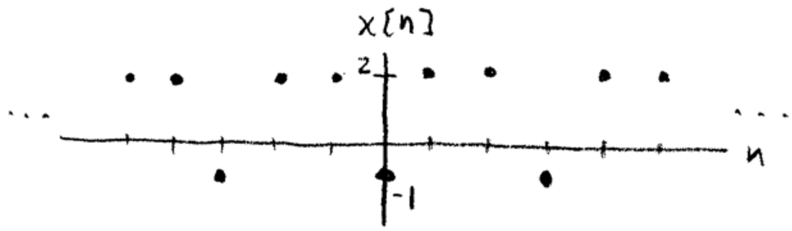
Consider the periodic sequence

$$a_k = \begin{cases} 1, & k = \ldots, -9, -6, -3, 0, 3, 6, 9, \ldots \\ -1, & \text{otherwise} \end{cases}$$

a) Find the DT signal x[n] (with fundamental period N=3) having Fourier coefficients a_k.

$$x[n] = \sum_{k=-1}^{1} a_k e^{jk\frac{2\pi}{3}n} = -e^{-j\frac{2\pi}{3}n} + 1 - e^{j\frac{2\pi}{3}n}$$

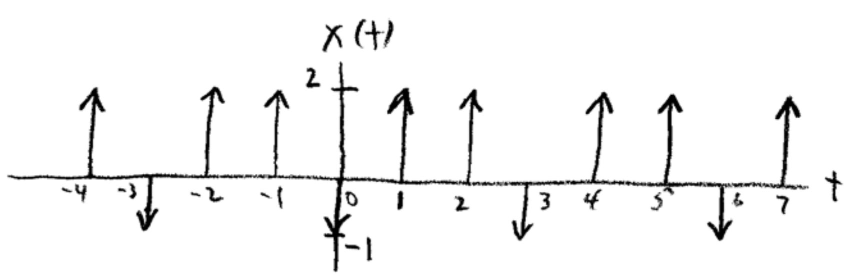
$$= 1 - 2\cos\frac{2\pi}{3}n = \ldots, -1, 2, 2, \underset{\substack{\uparrow \\ n=0}}{-1}, 2, 2, -1, \ldots$$



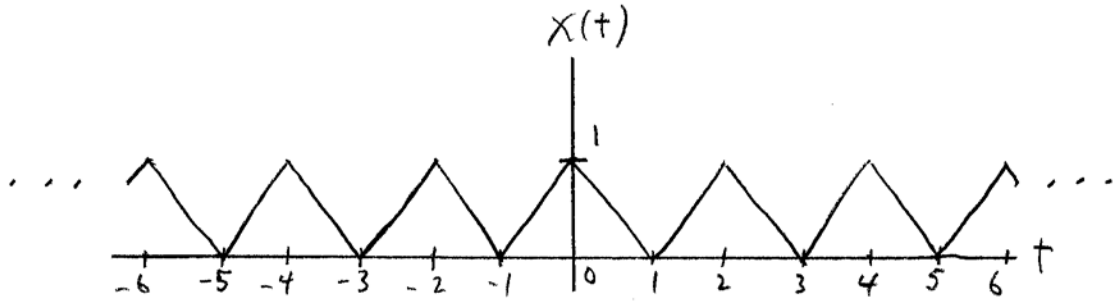
b) Find the CT signal x(t) (with fundamental period T=3) having Fourier coefficients a_k.

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\,\delta(t-n) = \sum_{n=-\infty}^{\infty} \left(1 - 2\cos\frac{2\pi}{3}n\right)\delta(t-n)$$

$$= \ldots - \delta(t+3) + 2\delta(t+2) + 2\delta(t+1) - \delta(t) + 2\delta(t-1) + 2\delta(t-2) - \delta(t-3) + \ldots$$



Consider the periodic CT signal x(t) shown below.



a) Find the Fourier coefficients a_k of x(t).

$$T = 2, \quad \omega_0 = \pi$$

from Problem 1

$$a_k = \frac{1}{2}\int_{-1}^{1} x(t) e^{-jk\pi t}\,dt = \frac{1}{2} X(k\pi)$$

$$= \begin{cases} \frac{1}{2}, & k = 0 \\ \frac{1-(-1)^k}{k^2\pi^2}, & k \neq 0 \end{cases}$$