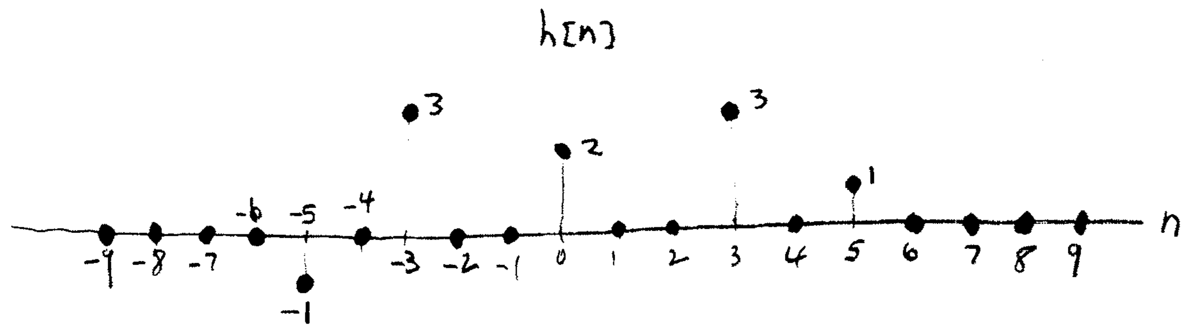
Consider the linear time-invariant system that has the discrete-time impulse response shown below. (Assume $h[n]=0$ for $|n|>9$).



a) Determine $H(\Omega)$.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\Omega n} = -e^{j5\Omega} + 3e^{j3\Omega} + 2 + 3e^{-j3\Omega} + e^{-j5\Omega}$$

$$= 2 + 6\cos 3\Omega - j2\sin 5\Omega$$

b) Suppose you have determined in part a) that

$$H(\Omega) = 2 + 4\sin(2\Omega) + 2\cos(6\Omega).$$

Find the output $y[n]$ corresponding to the input

$$x[n] = 4 + 8\cos(\tfrac{\pi}{2}n).$$

$$x[n] = 4e^{-j\frac{\pi}{2}n} + 4 + 4e^{j\frac{\pi}{2}n}, \quad N = 4, \quad a_{-1} = a_0 = a_1 = 4,$$

$$\Omega_0 = \frac{\pi}{2}, \quad a_2 = 0$$

$$y[n] = \sum_{k=-1}^{2} b_k e^{jk\frac{\pi}{2}n}, \quad b_k = a_k H(k\Omega_0)$$

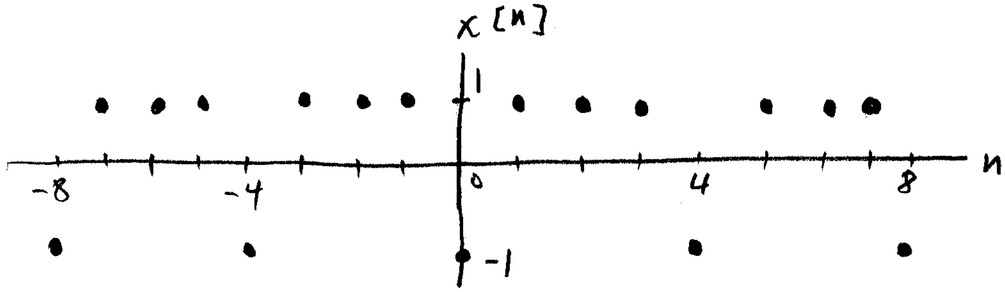
$$H(k\Omega_0) = 2 + 4\sin(k\pi) + 2\cos(3k\pi) = \begin{cases} 4, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$$

$$b_0 = 4, \quad b_{-1} = b_1 = b_2 = 0, \quad y[n] = 16$$

Consider the LTI system with impulse response

$$h[n] = (\tfrac{1}{2})^{|n|}$$

and input x[n] depicted below:



a) Find the Fourier coefficients of x[n].

$$N_0 = 4, \quad \Omega_0 = \frac{2\pi}{N_0} = \frac{\pi}{2}$$

$$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]\, e^{-jk\frac{\pi}{2}n} = \frac{1}{4}\left(-1 + e^{-jk\frac{\pi}{2}} + e^{-jk\pi} + e^{-jk\frac{3\pi}{2}}\right)$$

$$= \frac{1}{4}\left(-1 + (-j)^k + (-1)^k + j^k\right)$$

$$= \begin{cases} \frac{1}{2}; & k = \ldots, -8, -4, 0, 4, 8, \ldots \\ -\frac{1}{2}; & \text{otherwise} \end{cases}$$

b) Find the transfer function H(Ω) of the system.

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} (\tfrac{1}{2})^{|n|} e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{0} (\tfrac{1}{2})^{-n} e^{-j\Omega n} + \sum_{n=0}^{\infty} (\tfrac{1}{2})^{n} e^{-j\Omega n} - 1$$

$$= \sum_{m=0}^{\infty} (\tfrac{1}{2})^{m} e^{j\Omega m} + \sum_{n=0}^{\infty} (\tfrac{1}{2})^{n} e^{-j\Omega n} - 1$$

$$= \frac{1}{1 - \frac{1}{2}e^{j\Omega}} + \frac{1}{1 - \frac{1}{2}e^{-j\Omega}} - 1$$

$$= \frac{3}{5 - 4\cos\Omega}$$

c) Find the Fourier coefficients of the system output y[n].

$$b_k = H(k\Omega_0)a_k = \frac{3}{5-4\cos k\frac{\pi}{2}}a_k$$

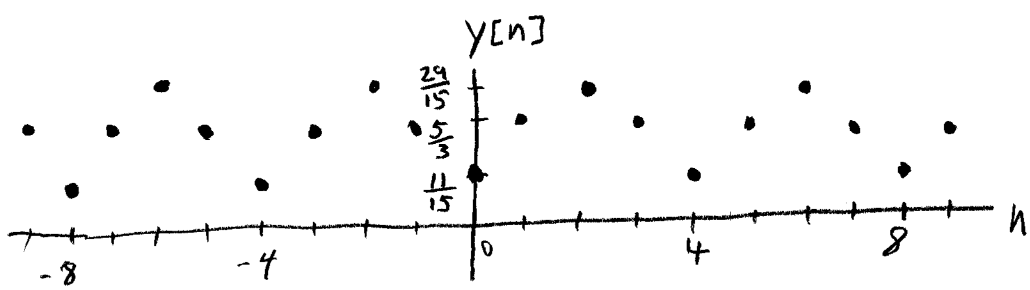
$$= \begin{cases} \frac{3}{2} ; & k = \ldots,-8,-4,0,4,8,\ldots \\ -\frac{3}{10} ; & k \text{ odd} \\ -\frac{1}{6} ; & \text{otherwise} \end{cases}$$

d) Draw a graph of the system output y[n] versus n.

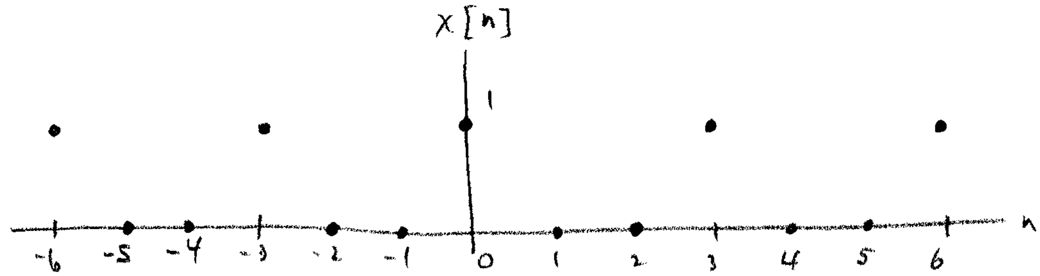
$$y[n] = \sum_{k=0}^{3} b_k e^{jk\frac{\pi}{2}n} = \frac{3}{2} - \frac{3}{10}e^{j\frac{\pi}{2}n} - \frac{1}{6}e^{j\pi n} - \frac{3}{10}e^{j\frac{3\pi}{2}n}$$

$$= \frac{3}{2} - \frac{3}{10}j^n - \frac{1}{6}(-1)^n - \frac{3}{10}(-j)^n$$

$$= \begin{cases} \frac{11}{15} ; & n=0 \\ \frac{5}{3} ; & n=1,3 \\ \frac{29}{15} ; & n=2 \end{cases}$$



Consider the LTI DT system with impulse response $h[n] = e^{-|n|}$. The input to the system x[n] is shown below.



a) Find the transfer function $H(\omega)$ of the system.

$$H(w) = \sum_{n=-\infty}^{\infty} e^{-|n|} e^{-jwn} = \sum_{n=-\infty}^{0} e^{(1-jw)n} + \sum_{n=0}^{\infty} e^{-(1+jw)n} - 1$$

$$= \sum_{n=0}^{\infty} e^{-(1-jw)n} + \sum_{n=0}^{\infty} e^{-(1+jw)n} - 1 = \frac{1}{1-e^{-(1-jw)}} + \frac{1}{1-e^{-(1+jw)}} - 1$$

$$= \frac{1-e^{-(1+jw)} + 1 - e^{-(1-jw)} - (1-e^{-(1-jw)})(1-e^{-(1+jw)})}{(1-e^{-(1-jw)})(1-e^{-(1+jw)})}$$

$$= \frac{1-e^{-2}}{1-2e^{-1}\cos w + e^{-2}}$$

b) Find the Fourier coefficients a_k of x[n].

$$a_k = \frac{1}{3}\sum_{n=0}^{2} x[n] e^{-jk\frac{2\pi}{3}n} = \frac{1}{3}$$

c) Find the Fourier coefficients b_k of the system output.

$$b_k = a_k H\left(k\frac{2\pi}{3}\right) = \frac{1}{3}\left(\frac{1-e^{-2}}{1-2e^{-1}\cos\left(\frac{2\pi}{3}k\right)+e^{-2}}\right)$$

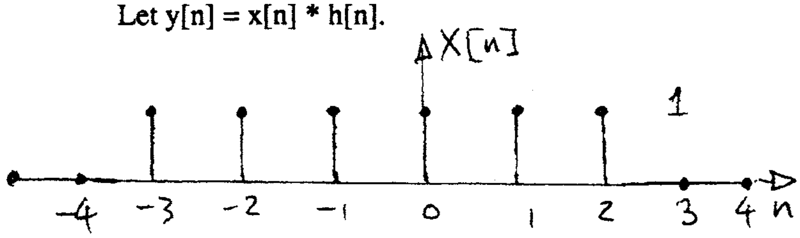
$$= \begin{cases} \frac{1}{3}\left(\frac{1-e^{-2}}{1+e^{-1}+e^{-2}}\right), & k=\pm 1 \\ \frac{1}{3}\left(\frac{1-e^{-2}}{1-2e^{-1}+e^{-2}}\right), & k=0 \end{cases} \quad \text{(period 3)}$$

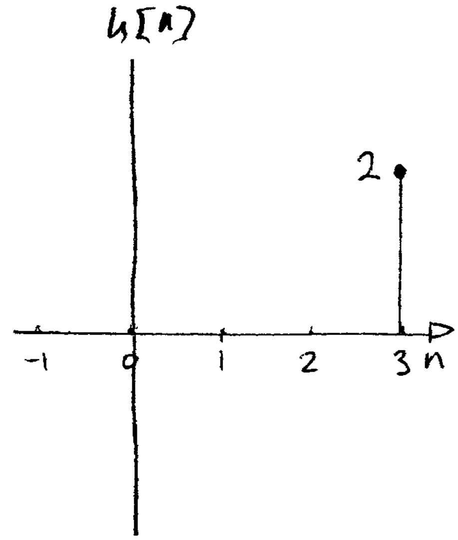
d) Find a time-domain description (graph or formula) of the output y[n].

$$y[n] = \sum_{k=-1}^{1} b_k e^{jk\frac{2\pi}{3}n} = \frac{1}{3}\left(1-e^{-2}\right)\left(\frac{e^{-j\frac{2\pi}{3}n}+e^{j\frac{2\pi}{3}n}}{1+e^{-1}+e^{-2}} + \frac{1}{1-2e^{-1}+e^{-2}}\right)$$

$$= \frac{1}{3}\left(1-e^{-2}\right)\left(\frac{2\cos\frac{2\pi}{3}n}{1+e^{-1}+e^{-2}} + \frac{1}{1-2e^{-1}+e^{-2}}\right)$$

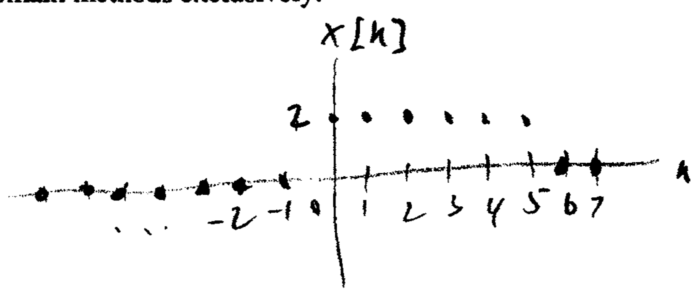
Consider the DT signals x[n] and h[n] shown below.
Let y[n] = x[n] * h[n].





a) Find y[n] by using time-domain methods exclusively.

$y[n] = 2x[n-3]$



b) Find y[n] by using frequency-domain methods exclusively.

$$X(\Omega) = 2\sum_{n=-3}^{2} e^{-j\Omega n}$$

$$H(\Omega) = e^{-j3\Omega}$$

$$Y(\Omega) = 2\sum_{n=0}^{5} e^{-j\Omega n}$$

$$y[n] = \begin{cases} 2, & n = 0, \dots, 5 \\ 0, & \text{else.} \end{cases}$$

Find the Laplace transform of each of the following signals. In each case, identify the region of convergence.

a) $x(t) = e^t u(-t)$

$$X(s) = \int_{-\infty}^{\infty} e^t u(-t) e^{-st} dt = \int_{-\infty}^{0} e^{(1-s)t} dt = \frac{1}{1-s} e^{(1-s)t} \Big|_{-\infty}^{0}$$

ROC: $\mathrm{Re}(1-s) > 0 \iff \mathrm{Re}\, s < 1$

$$X(s) = \frac{1}{1-s}(1-0) = \frac{1}{1-s}$$

b) $x(t) = \begin{cases} \sin t, & |t| \le \pi \\ 0, & |t| > \pi \end{cases}$

$$X(s) = \int_{-\pi}^{\pi} \sin t\, e^{-st} dt = \frac{1}{2j}\left(\int_{-\pi}^{\pi} e^{(j-s)t} dt - \int_{-\pi}^{\pi} e^{-(j+s)t} dt \right)$$

$$= \frac{1}{2j}\left(\frac{1}{j-s} e^{(j-s)t} \Big|_{-\pi}^{\pi} + \frac{1}{j+s} e^{-(j+s)t} \Big|_{-\pi}^{\pi} \right)$$

$$= \frac{1}{2j}\left(\frac{1}{j-s}\left(e^{(j-s)\pi} - e^{-(j-s)\pi}\right) + \frac{1}{j+s}\left(e^{-(j+s)\pi} - e^{(j+s)\pi}\right) \right)$$

$$= \frac{1}{2j}\left(\frac{1}{j-s}\left(-e^{-s\pi} + e^{s\pi}\right) + \frac{1}{j+s}\left(-e^{-s\pi} + e^{s\pi}\right) \right)$$

$$= \frac{1}{2j}\left(\left(e^{s\pi} - e^{-s\pi}\right)\left(\frac{1}{j-s} + \frac{1}{j+s} \right) \right)$$

$$= \frac{e^{-s\pi} - e^{s\pi}}{s^2+1}$$

ROC: entire plane

c) $x(t) = \sum_{n=0}^{\infty} (-1)^n \delta(t-n)$

$$X(s) = \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} (-1)^n \delta(t-n) e^{-st} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \delta(t-n) e^{-st} dt$$

$$= \sum_{n=0}^{\infty} (-1)^n e^{-sn} \int_{-\infty}^{\infty} \delta(t-n) dt$$

$$= \sum_{n=0}^{\infty} (-e^{-s})^n$$

ROC: $|-e^{-s}| < 1 \iff e^{-\text{Re}\, s} < 1 \iff \text{Re}\, s > 0$

$$X(s) = \frac{1}{1+e^{-s}}$$

Consider the LTI system governed by the differential equation

$$\ddot{y} + y = \ddot{x} - 2\dot{x} + x$$

and initial conditions

$$y(0^-) = \dot{y}(0^-) = 0$$

(i.e. assume the system is initially at rest). Let the input function be

$$x(t) = tu(t).$$

a) Find the transfer function H(s) of the system.

$$H(s) = \frac{s^2 - 2s + 1}{s^2 + 1}$$

b) Find the Laplace transform X(s) of the input x(t).

$$X(s) = \frac{1}{s^2}$$

c) Find the Laplace transform Y(s) of the output y(t).

$$Y(s) = H(s)X(s) = \frac{s^2-2s+1}{s^2(s^2+1)}$$

d) Find y(t).

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+j} + \frac{D}{s-j}$$

$$B = s^2 Y(s)\Big|_{s=0} = \frac{s^2-2s+1}{s^2+1}\Bigg|_{s=0} = 1$$

$$C = (s+j)Y(s)\Big|_{s=-j} = \frac{s^2-2s+1}{s^2(s-j)}\Bigg|_{s=-j} = 1$$

$$D = C^* = 1$$

$$\frac{A}{s} = \frac{s^2-2s+1}{s^2(s^2+1)} - \frac{1}{s^2} - \frac{1}{s+j} - \frac{1}{s-j} = \frac{-2s(s^2+1)}{s^2(s^2+1)} = -\frac{2}{s}$$

$$A = -2$$

$$y(t) = (-2+t+e^{jt}+e^{-jt})u(t) = (-2+t+2\cos t)u(t)$$

Using partial fraction expansion techniques, find the impulse response of the LTI system governed by the differential equation

$$y^{(4)} + 4y^{(3)} + 7\ddot{y} + 6\dot{y} + 2y = 3\ddot{x} + 6\dot{x} + 4x.$$

(**Hint:** $s^4+4s^3+7s^2+6s+2 = (s+1)^2(s^2+2s+2)$)

$$H(s) = \frac{3s^2+6s+4}{s^4+4s^3+7s^2+6s+2} = \frac{A_{11}}{s+1} + \frac{A_{12}}{(s+1)^2} + \frac{A_{21}}{s+1+j} + \frac{A_{31}}{s+1-j}$$

$$A_{12} = (s+1)^2 H(s)\Big|_{s=-1} = \frac{3s^2+6s+4}{s^2+2s+2}\Big|_{s=-1} = 1$$

$$A_{21} = (s+1+j)H(s)\Big|_{s=-1-j} = \frac{3s^2+6s+4}{(s+1)^2(s+1-j)}\Big|_{s=-1-j} = j$$

$$A_{31} = A_{21}^* = -j$$

$$H(s) - \frac{A_{12}}{(s+1)^2} - \frac{A_{21}}{s+1+j} - \frac{A_{31}}{s+1-j}$$

$$= \frac{3s^2+6s+4-(s^2+2s+2)-j(s+1)^2(s+1-j)+j(s+1)^2(s+1+j)}{(s+1)^2(s^2+2s+2)}$$

$$= \frac{2s^2+4s+2-2(s^2+2s+1)}{(s+1)^2(s^2+2s+2)} = 0$$

$$A_{11} = 0$$

$$h(t) = \left(te^{-t} + j\,e^{-(1+j)t} - j\,e^{-(1-j)t}\right)u(t)$$

$$= (t + 2\sin t)e^{-t}u(t)$$

Consider the causal LTI system governed by the differential equation

$$\dddot{y}+2\ddot{y}+\dot{y}=\ddot{x}+x\,.$$

a) Find the transfer function $H(s)$ of the system.

$$H(s)=\frac{s^2+1}{s^3+2s^2+s}$$

b) Determine whether the system is BIBO stable.

$s^3+2s^2+s = s(s+1)^2$

poles: $0, -1, -1$ not BIBO stable

c) Using partial fraction expansion, find the impulse response $h(t)$ of the system.

$$H(s) = \frac{A_{11}}{s} + \left(\frac{A_{21}}{s+1} + \frac{A_{22}}{(s+1)^2}\right)$$

$$A_{11} = sH(s)\Big|_{s=0} = \frac{s^2+1}{(s+1)^2}\Big|_{s=0} = 1$$

$$A_{22} = (s+1)^2H(s)\Big|_{s=-1} = \frac{s^2+1}{s}\Big|_{s=-1} = -2$$

$$H(s) - \frac{A_{11}}{s} - \frac{A_{22}}{(s+1)^2} = \frac{s^2+1-(s^2+2s+1)+2s}{s(s+1)^2} = 0 = \frac{A_{21}}{s+1}$$

$A_{21} = 0$

$$H(s) = \frac{1}{s} - \frac{2}{(s+1)^2}$$

$$h(t) = u(t) - 2te^{-t}u(t)$$

Using partial fraction expansion, find the impulse response of the causal LTI system described by the differential equation

$$y^{(4)} + \ddot{y} = x.$$

$$H(s) = \frac{1}{s^4+s^2} = \frac{1}{s^2(s+j)(s-j)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+j} + \frac{D}{s-j}$$

$$B = s^2 H(s)\Big|_{s=0} = \frac{1}{(s+j)(s-j)}\Big|_{s=0} = 1$$

$$C = (s+j)H(s)\Big|_{s=-j} = \frac{1}{s^2(s-j)}\Big|_{s=-j} = -\frac{j}{2}$$

$$D = (s-j)H(s)\Big|_{s=j} = \frac{1}{s^2(s+j)}\Big|_{s=j} = \frac{j}{2}$$

$$H(s) - \frac{B}{s^2} - \frac{C}{s+j} + \frac{D}{s-j} = \frac{1-(s+j)(s-j)+\frac{j}{2}s^2(s-j)-\frac{j}{2}(s+j)}{s^2(s+j)(s-j)}$$

$$= 0$$

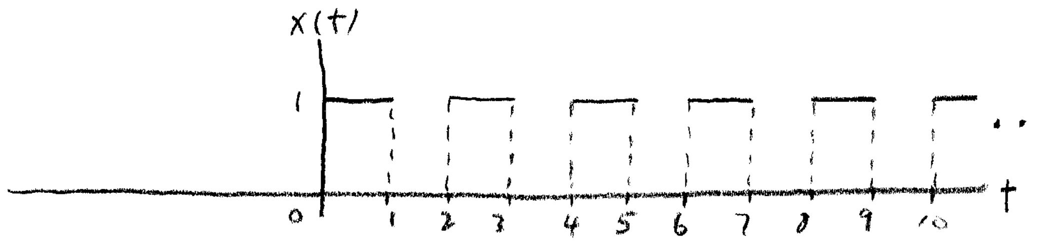
$$A = 0$$

$$H(s) = \frac{1}{s^2} - \frac{j/2}{s+j} + \frac{j/2}{s-j}$$

$$h(t) = tu(t) - \frac{j}{2}e^{-jt}u(t) + \frac{j}{2}e^{jt}u(t)$$

$$= (t - \sin t)u(t)$$

Let x(t) be the CT signal shown. Let X(s) be the Laplace transform of x(t).



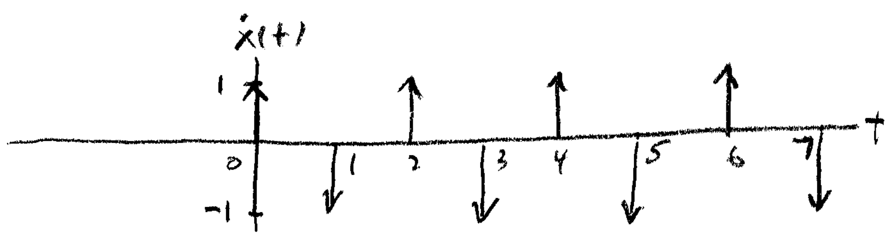
a) Find X(s). (Be sure to identify the ROC.)

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \sum_{k=0}^{\infty}\int_{2k}^{2k+1} e^{-st}dt = \sum_{k=0}^{\infty}\left(-\frac{1}{s}\left(e^{-(2k+1)s} - e^{-2ks}\right)\right)$$

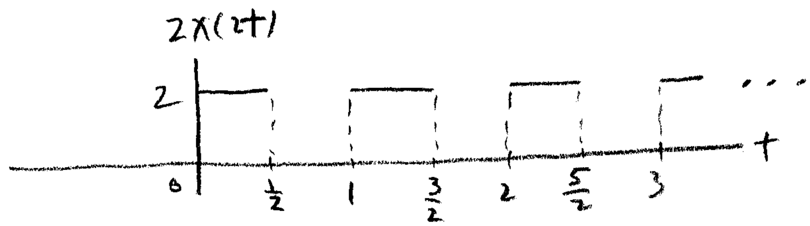
$$= \frac{1-e^{-s}}{s}\sum_{k=0}^{\infty} e^{-2ks} = \frac{1-e^{-s}}{s(1-e^{-2s})} = \frac{1}{s(1+e^{-s})}$$

The series converges for Re s > 0.

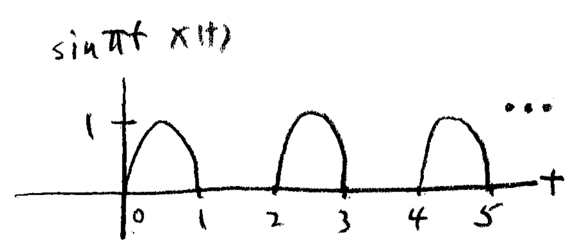
b) Draw a graph of $\mathcal{L}^{-1}\{sX(s)\}$. $\dot{x}(t) \Leftrightarrow sX(s)$



c) Draw a graph of $\mathcal{L}^{-1}\{X(\frac{s}{2})\}$. $2x(2t) \Leftrightarrow X(\frac{s}{2})$



d) Draw a graph of $\mathcal{L}^{-1}\{\frac{X(s-j\pi)-X(s+j\pi)}{2j}\}$.



$e^{j\pi}x(t) \Leftrightarrow X(s-j\pi)$

$e^{-j\pi}x(t) \Leftrightarrow X(s+j\pi)$

$$\sin \pi t\, x(t) \Leftrightarrow \frac{X(s-j\pi)-X(s+j\pi)}{2j}$$