ECE 332
Homework #5

1) For each of the following transfer functions $G(s)$, sketch the root locus.

a) $\frac{1}{s(s+1)^3}$
b) $\frac{(2s+1)(3s+1)}{s^3}$
c) $\frac{s}{s^2+2s+2}$

2) For each plant $G(s)$ in problem 1), consider the corresponding gain compensated system. In each case, use Routh-Hurwitz methods to find the number of open RHP poles in the closed-loop system as a function of the gain K.

3) Repeat problem 2), using Nyquist plots. Compare your results with those obtained in 2).

4) Consider the plant and compensator

$$G(s) = \frac{1}{s^2-1}, \qquad G_c(s) = \frac{2s+2}{s+3}.$$

Investigate the effect of varying each compensator pole and zero as well as the compensator high-frequency gain on stability of the closed-loop system. That is, draw the root locus corresponding to each of the following parametrized compensators.

a) $G_c(s) = 2\frac{s+z}{s+3}$

b) $G_c(s) = 2\frac{s+1}{s+p}$

c) $G_c\,(s) = K\frac{s+1}{s+3}$

(Assume K, z, and p to be real.) For each system, determine the range of K, z, or p for closed-loop stability.

5) Consider the plant

$$G(s) = \frac{1}{s(s+1)}.$$

a) Using the formulas for second-order systems, design a gain K so that the gain compensated closed-loop system satisfies the specifications

i) $e_{ss1} \leq .5\ s$,
ii) $\omega_b \geq 2.5\ rad/s$,
iii) $M_p \leq 1.35$.

b) In MATLAB, type "$k = K; hw55$" to plot closed-loop frequency and step responses for the value of K obtained in part a). Verify that specifications ii) and iii) are met.

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Solutions

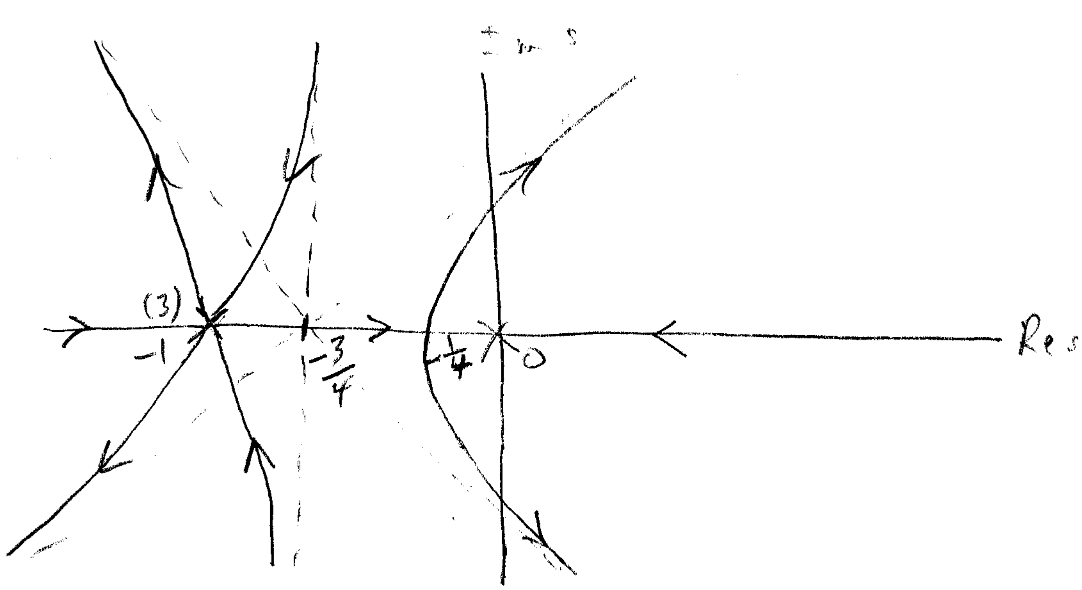
1a) $G(s) = \frac{1}{s(s+1)^3} = \frac{1}{s^4+3s^3+3s^2+s}$

saddle points:

$$N'D - ND' = -(4s^3+9s^2+6s+1)$$
$$= -4(s+1)^2(s+\tfrac{1}{4})$$
$$= 0$$
$$s = -1, -1, -\tfrac{1}{4}$$

asymptotes: $\sigma_c = -\frac{3}{4-0} = -\frac{3}{4}$

$$\theta_i = \frac{180i^\circ}{4-0} = 45i^\circ$$



b) $G(s) = \dfrac{(2s+1)(3s+1)}{s^3} = \dfrac{6s^2+5s+1}{s^3}$

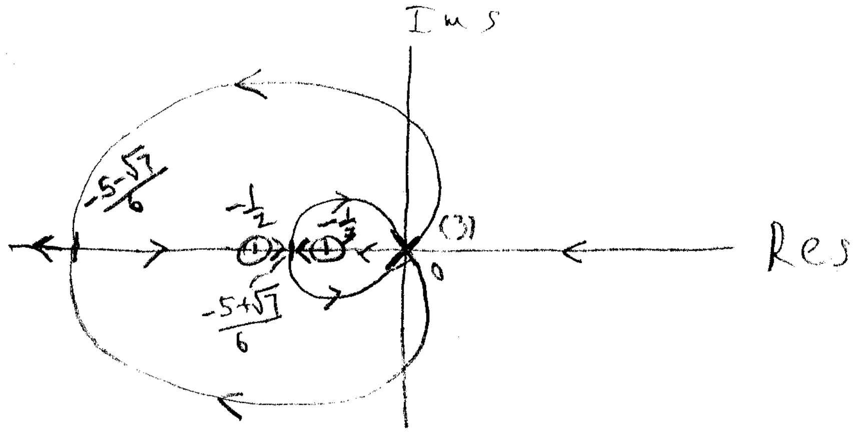
saddle points: $N'D - ND' = (12s+5)s^3 - (6s^2+5s+1)(3s^2)$

$= -6s^4 - 10s^3 - 3s^2$

$= -6s^2(s^2 + \frac{5}{3}s + \frac{1}{2})$

$= 0$

$s = \dfrac{-5 \pm \sqrt{7}}{6}, 0, 0$



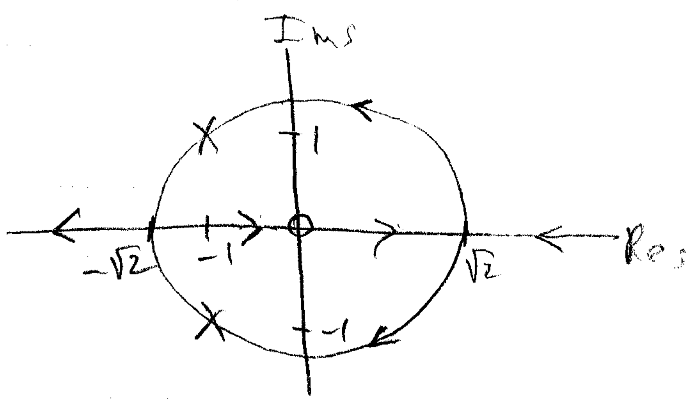
c) $G(s) = \dfrac{s}{s^2+2s+2}$

saddle points: $N'D - ND' = s^2+2s+2 - s(2s+2)$

$= -s^2+2$

$= 0$

$s = \pm\sqrt{2}$



2a) $D+KN = s(s+1)^3 + K = s^4+3s^3+3s^2+s+K$

Routh table:

$$\begin{array}{ccc} 1 & 3 & K \\ 3 & 1 & 0 \\ \frac{8}{3} & K & 0 \\ 1-\frac{9}{8}K & 0 & \\ K & & \end{array}$$

$1-\frac{9}{8}K > 0 \iff K < \frac{8}{9}$

# of open RHP poles = # of 1st column sign change.

$$= \begin{cases} 0, & 0 < K < \frac{8}{9} \\ 1, & K < 0 \\ 2, & K > \frac{8}{9} \end{cases}$$

b) $D+KN = s^3 + K(6s^2+5s+1) = s^3+6Ks^2+5Ks+K$

Hurwitz matrix:

$$\mathcal{H} = \begin{bmatrix} 6K & K & 0 \\ 1 & 5K & 0 \\ 0 & 6K & K \end{bmatrix}$$

$$m_1 = 6K$$

$$m_2 = \begin{vmatrix} 6K & K \\ 1 & 5K \end{vmatrix} = (30K-1)K$$

$$m_3 = \begin{vmatrix} 6K & K & 0 \\ 1 & 5K & 0 \\ 0 & 6K & K \end{vmatrix} = (30K-1)K^2$$

$m_1 > 0 \iff K > 0$

$m_2 > 0 \iff K < 0$ or $K > \frac{1}{30}$

$m_3 > 0 \iff K > \frac{1}{30}$

\# of open RHP poles = total # of sign changes in $1, m_1, m_3$ and $1, m_2$

$$= \begin{cases} 0, & K > \frac{1}{30} \\ 1, & K < 0 \\ 2, & 0 < K < \frac{1}{30} \end{cases}$$

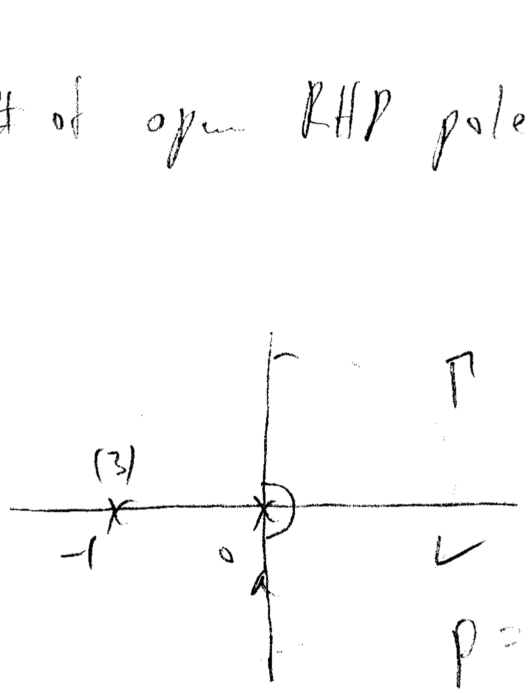
c) $D + KN = s^2 + 2s + 2 + Ks = s^2 + (2+K)s + 2$

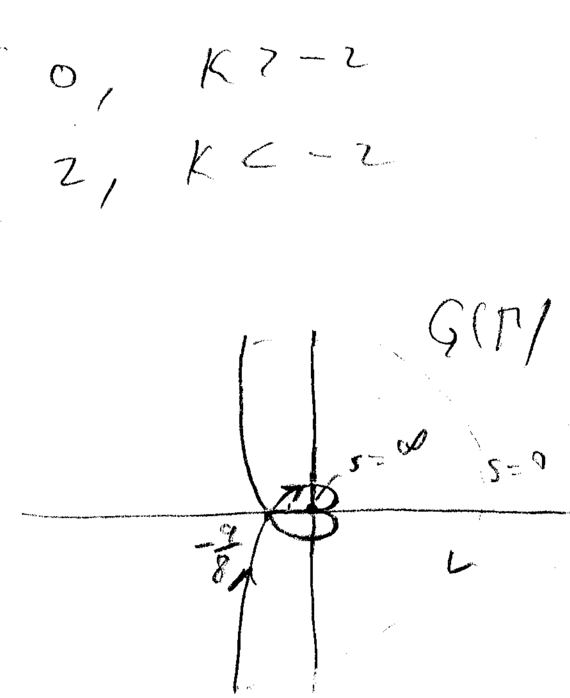
$$\mathcal{H} = \begin{bmatrix} 2+K & 0 \\ 1 & 2 \end{bmatrix}$$

$$m_1 = 2 + K$$
$$m_2 = 2(2+K)$$

$$\#\text{ of open RHP poles} = \begin{cases} 0, & K > -2 \\ 2, & K < -2 \end{cases}$$

3a)





axis crossing: $\angle G(j\omega) = \angle \dfrac{1}{j\omega(j\omega+1)^3}$

$$= -90° - 3\angle j\omega + 1$$

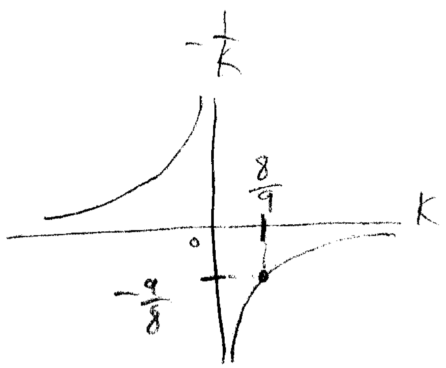
$$= -180°$$

$$\angle j\omega + 1 = \frac{90}{3} = 30°$$

$$\omega_p = \tan 30° = \frac{1}{\sqrt{3}}$$

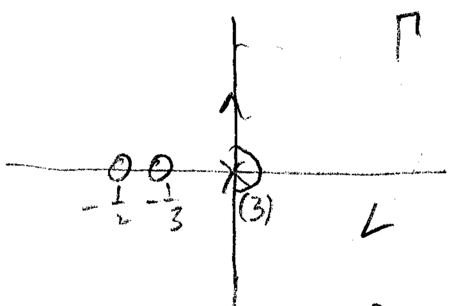
$$G\left(j\tfrac{1}{\sqrt{3}}\right) = -\left|\frac{1}{j\frac{1}{\sqrt{3}}\left(j\frac{1}{\sqrt{3}}+1\right)^3}\right| = -\frac{9}{8}$$

$$\gamma = \begin{cases} 0, & -\frac{1}{K} < -\frac{9}{8} \\ -2, & -\frac{9}{8} < -\frac{1}{K} < 0 \\ -1, & -\frac{1}{K} > 0 \end{cases}$$

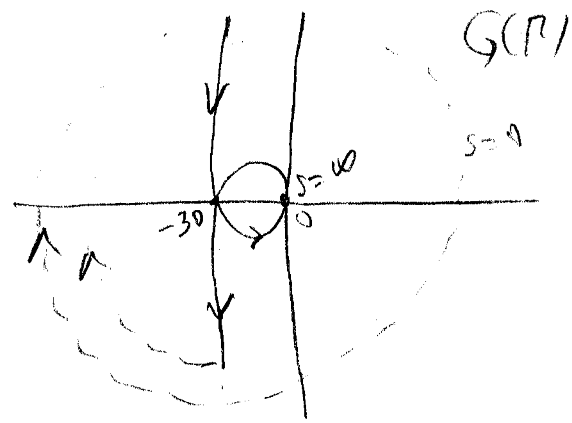


$$p - \gamma = \begin{cases} 0, & 0 < K < \frac{8}{9} \\ 1, & K < 0 \\ 2, & K > \frac{8}{9} \end{cases}$$

b)



$p = 0$



axis crossing:

$$\angle G(j\omega) = \angle \frac{(j2\omega+1)(j3\omega+1)}{(j\omega)^3}$$

$$= -270^\circ + \angle -6\omega^2 + j5\omega + 1$$

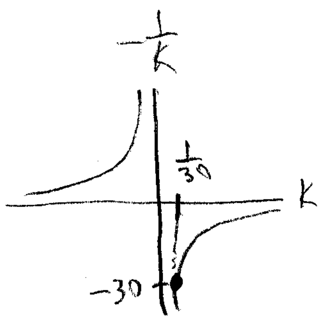
$$= -180^\circ$$

$$\angle 1 - 6\omega^2 + j5\omega = 90^\circ$$

$$1 - 6\omega^2 = 0$$

$$\omega_p = \frac{1}{\sqrt{6}}$$

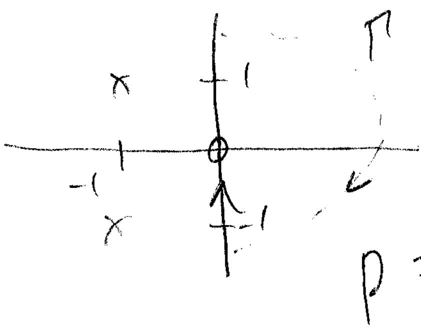
$$G\left(j\frac{1}{\sqrt{6}}\right) = \frac{\left(j\frac{2}{\sqrt{6}}+1\right)\left(j\frac{3}{\sqrt{6}}+1\right)}{\left(j\frac{1}{\sqrt{6}}\right)^3} = \frac{j\frac{5}{\sqrt{6}}}{\left(j\frac{1}{\sqrt{6}}\right)^3} = -30$$



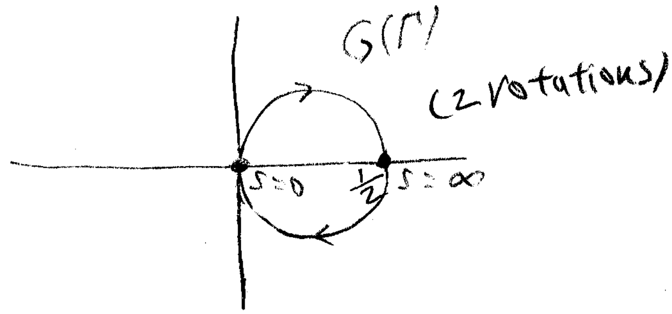
$$\gamma = \begin{cases} -2, & -\frac{1}{k} < -30 \\ 0, & -30 < -\frac{1}{k} < 0 \\ -1, & -\frac{1}{k} > 0 \end{cases}$$

$$p - \gamma = \begin{cases} 0, & k > \frac{1}{30} \\ 1, & k < 0 \\ 2, & 0 < k < \frac{1}{30} \end{cases}$$

c)



$p = 0$



axis crossing: $\angle G(j\omega) = \angle \frac{j\omega}{(j\omega)^2 + j2\omega + 2}$

$$= 90^\circ - \angle\left(2 - \omega^2 + j2\omega\right)$$

$$= 0$$

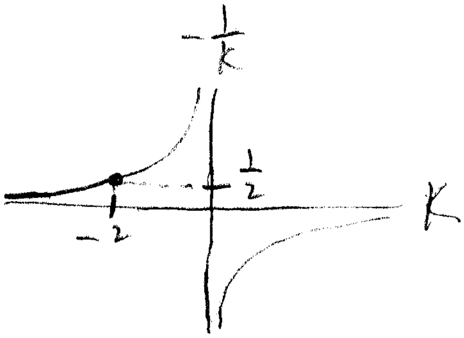
$$\angle 2 - w^2 + j2w = 90^\circ$$

$$2 - w^2 = 0$$

$$w_p = \sqrt{2}$$

$$G(j\sqrt{2}) = \frac{j\sqrt{2}}{(j\sqrt{2})^2 + j2\sqrt{2} + 2} = \frac{1}{2}$$

$$\gamma = \begin{cases} 0, & -\frac{1}{K} < 0 \text{ or } -\frac{1}{K} > \frac{1}{2} \\ -2, & 0 < -\frac{1}{K} < \frac{1}{2} \end{cases}$$



$$p - \gamma = \begin{cases} 0, & K > -2 \\ 2, & K < -2 \end{cases}$$

4a) Let $G = \frac{N}{D}$, $G_c = \frac{N_c}{D_c}$.

$$H = \frac{G_c G}{1 + G_c G} = \frac{N_c N}{D_c D + N_c N}$$

$$D_c D + N_c N = (s+3)(s^2-1) + 2(s+z)$$

$$= s^3 + 3s^2 + s - 3 + 2z$$

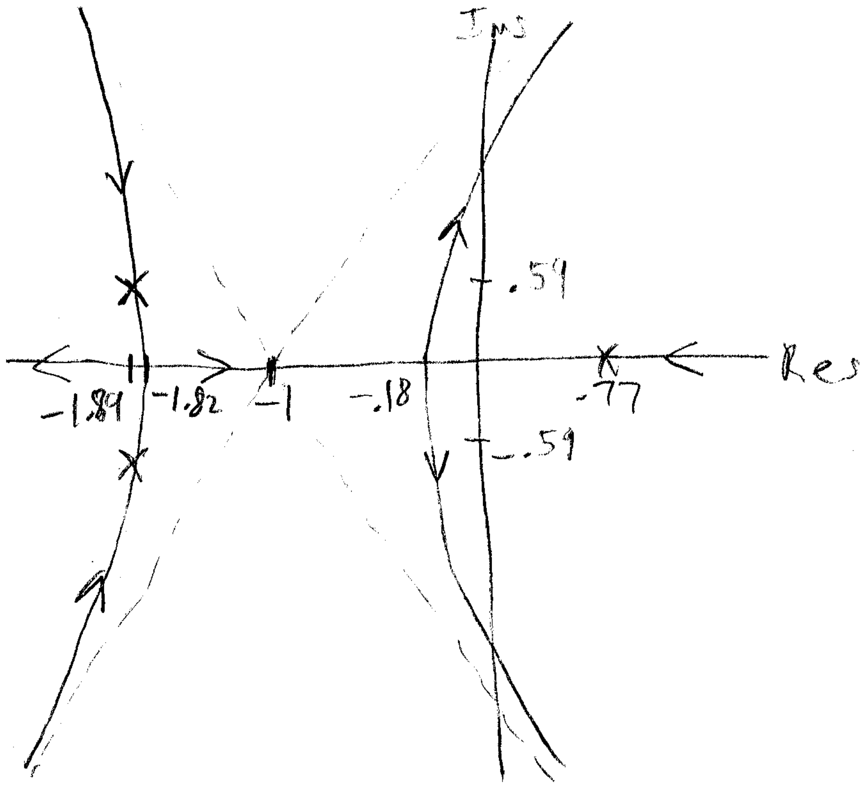
$$\bar{D} = s^3 + 3s^2 + s - 3 = (s - .77)(s + 1.89 + j.59)(s + 1.89 - j.59)$$

$$\bar{N} = 2$$

saddle points: $2(3s^2 + 6s + 1) = 0 \Rightarrow s = -1.82, -.18$

asymptotes: $\sigma_c = -\frac{3}{3-0} = -1$

$$\theta_i = \frac{180 i^\circ}{3-0} = 60 i^\circ$$



$$\begin{array}{ccc} 1 & 1 & 0 \\ 3 & 2z-3 & 0 \\ 2-\frac{2}{3}z & 0 & \\ 2z-3 & & \end{array}$$

H stable for

$\frac{3}{2} < z < 3$

b) $D_c D + N_c N = (s+p)(s^2-1) + 2(s+1)$

$= (s^2 - s + 2 + p(s-1))(s+1)$

$\bar{D} = s^2 - s + 2 = (s - \frac{1}{2} + j\frac{\sqrt{7}}{2})(s - \frac{1}{2} - j\frac{\sqrt{7}}{2})$

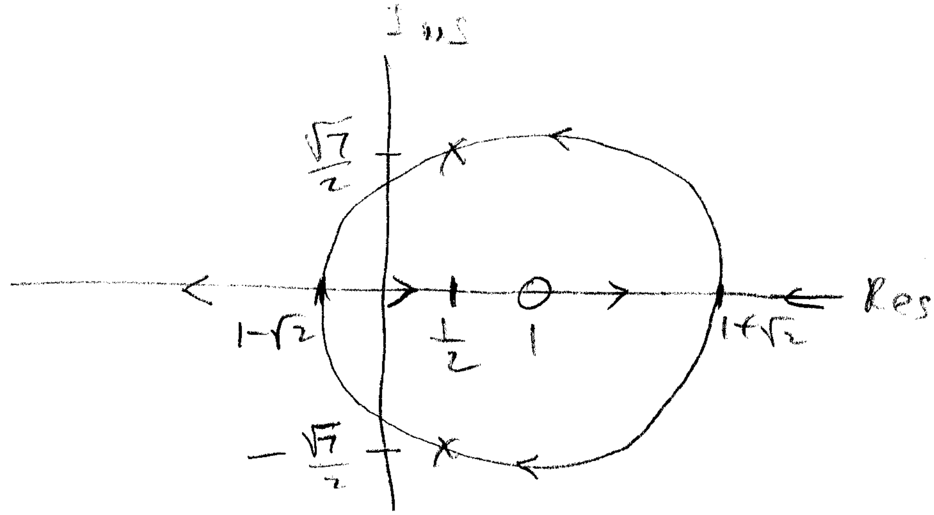
$\bar{N} = s - 1$

saddle points:

$s^2 - s + 2 - (s-1)(2s-1) = 0$

$s^2 - 2s - 1 = 0$

$s = 1 \pm \sqrt{2}$



A stable $\iff p-1>0$ and $2-p>0$.

$\iff 1<p<2$

c) $D_c D + N_c N = (s+3)(s^2-1) + K(s+1)$

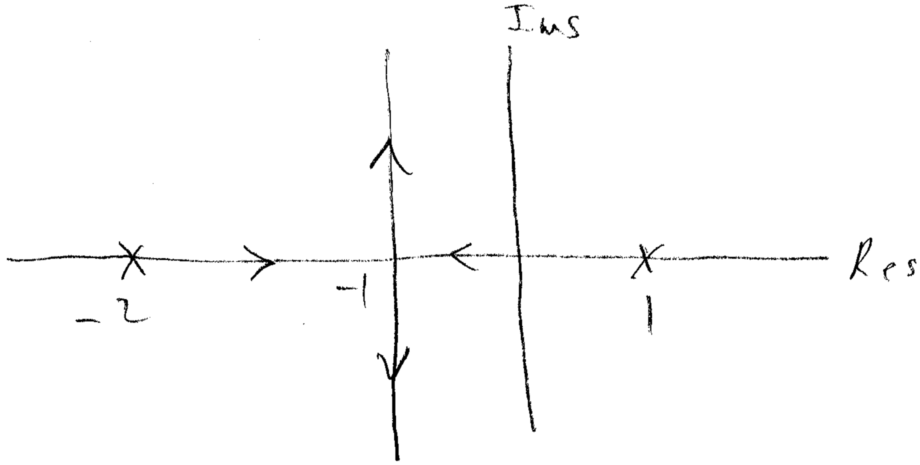
$= (s^2+2s-3+K)(s+1)$

$\bar{D} = s^2+2s-3 = (s+3)(s-1)$

$\bar{N} = 1$

Saddle points: $2s+2=0$

$s=-1$



H stabl $\iff$ $K > 3$

5a) $H = \dfrac{KG}{1+KG} = \dfrac{K}{s^2+s+K}$

$\omega_n^2 = K, \quad 2\xi\omega_n = 1$

$\omega_n = \sqrt{K}, \quad \xi = \dfrac{1}{2\sqrt{K}}$

$e_{ss1} = 2\dfrac{\xi}{\omega_n} = \dfrac{1}{K}$

$$\omega_b = \omega_n\sqrt{1-2\xi^2+2\sqrt{1-\xi^2+\xi^4}}$$

$$= \sqrt{K\left(1-\frac{1}{2K}+2\sqrt{1-\frac{1}{4K}+\frac{1}{16K^2}}\right)}$$

$$= \sqrt{K-\frac{1}{2}+\sqrt{4K^2-K+\frac{1}{4}}}$$

$$M_p = 1+e^{-\frac{\pi\xi}{\sqrt{1-\xi^2}}} = 1+e^{-\frac{\pi}{\sqrt{4K-1}}}$$

$e_{ss1} = .5 \Rightarrow K = 2$

$K = 2$ yields $M_p = 1 + e^{-\frac{\pi}{\sqrt{7}}} = 1.3$, but

$\omega_b = \sqrt{.8 + \sqrt{5.71}} = 1.79$. We need to increase

K to meet the ω_b specification. However,

increasing K too much will violate the M_p

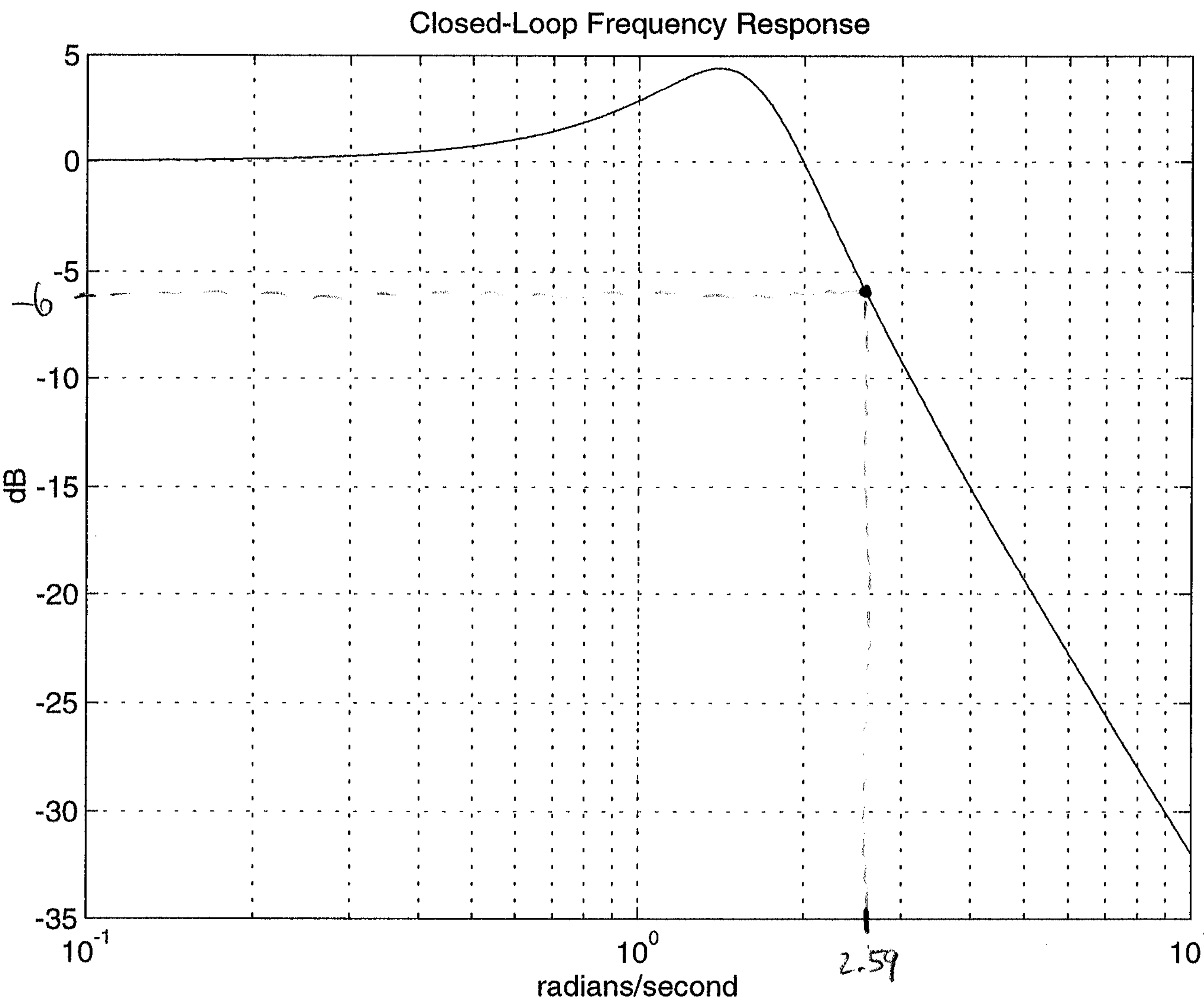
specification. The maximum admissible K is

obtained by solving for K in terms of M_p:

$$K = \frac{1}{4}\left(1 + \left(\frac{\pi}{\ln \frac{1}{M_p - 1}}\right)^2\right) = \frac{1}{4}\left(1 + \left(\frac{\pi}{\ln 2.86}\right)^2\right) = 2.48$$

This yields $e_{ss} = .4$, $\omega_b = 2.59$.

b)


Closed-Loop Frequency Response
5
0
-5
-6
-10
-15
-20
-25
-30
-35
dB
10^-1
10^0
2.59
10^1
radians/second

Closed-Loop Step Response