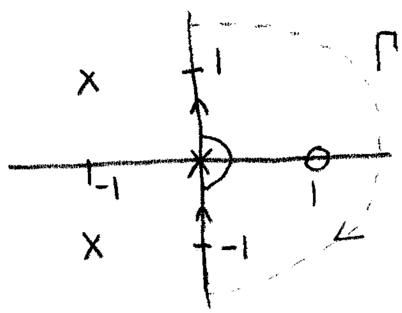
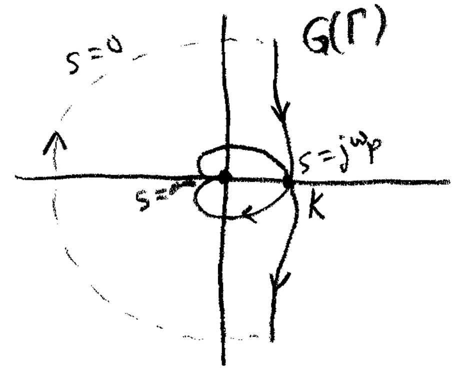
1) Consider the transfer function

$$G(s) = \frac{s-1}{s(s^2+2s+2)}.$$

a) Sketch the Nyquist plot for $G(s)$. Identify the location of all real-axis crossings.

$$s^2+2s+2 = (s+1+j)(s+1-j)$$





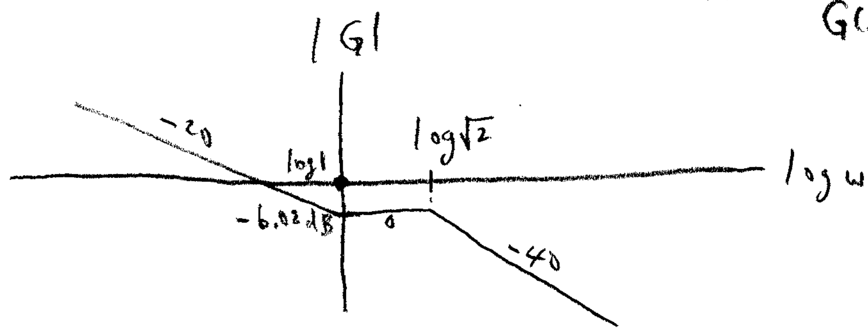
$$\angle \frac{j\omega_p - 1}{j2\omega_p + 2 - \omega_p^2} = 90^\circ$$

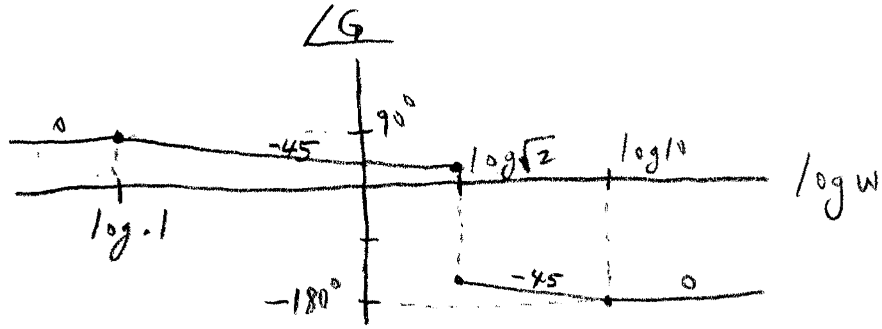
$$Re\left((j\omega_p - 1)(-j2\omega_p + 2 - \omega_p^2)\right) = 3\omega_p^2 - 2 = 0 \Rightarrow \omega_p = \sqrt{\tfrac{2}{3}}$$

$$K = G\left(j\sqrt{\tfrac{2}{3}}\right) = \frac{j\sqrt{\frac{2}{3}} - 1}{j\sqrt{\frac{2}{3}}\left(-\frac{2}{3} + j2\sqrt{\frac{2}{3}} + 2\right)} = \frac{j\sqrt{\frac{2}{3}} - 1}{\frac{4}{3}\left(j\sqrt{\frac{2}{3}} - 1\right)} = \frac{3}{4}$$

b) Draw the straight-line approximations to the Bode plots for $G(s)$.

$$G(s) = -\frac{1}{2}\,\frac{\frac{s}{-1}+1}{s\left(\frac{s^2}{2}+s+1\right)}$$





2) Use the euclidean algorithm and either the Routh table or the Hurwitz matrix to determine whether the transfer function

$$G(s) = \frac{s^2+s-2}{s^5+s^4+s^3-s^2-s-1} = \frac{\Omega(s)}{\Delta(s)}$$

is BIBO stable.

Euclidean algorithm:

Divide Ω by Δ. $Q_1 = 0$, $R_1 = \Omega$

Divide Δ by R_1.

$$\begin{array}{cc|cccccc} -1 & 2 & 1 & 1 & 1 & -1 & -1 & -1 \\ & & & & 2 & 0 & 6 & -8 \\ & & & -1 & 0 & -3 & 4 & \\ \hline & & 1 & 0 & 3 & -4 & 9 & -9 \end{array}$$

$Q_2 = s^3+3s-4$

$R_2 = 9s-9$

Divide R_1 by $\frac{R_2}{9}$.

$$\begin{array}{c|ccc} 1 & 1 & 1 & -2 \\ & & 1 & 2 \\ \hline & 1 & 2 & 0 \end{array}$$

$Q_3 = s+2$

$R_3 = 0$

Ω, Δ are not coprime. $GCD = \frac{R_2}{9} = s-1$

$$\frac{\Omega}{s-1} = Q_3 = s+2$$

$$\begin{array}{c|cccccc} 1 & 1 & 1 & 1 & -1 & -1 & -1 \\ & & 1 & 2 & 3 & 2 & 1 \\ \hline & 1 & 2 & 3 & 2 & 1 & 0 \end{array}$$

$$\frac{\Delta}{s-1} = s^4+2s^3+3s^2+2s+1$$

$$G(s) = \frac{s+2}{s^4+2s^3+3s^2+2s+1} = \frac{N(s)}{D(s)}$$

Routh table:

$$\begin{array}{ccc} 1 & 3 & 1 \\ 2 & 2 & 0 \\ 2 & 1 & \\ 1 & 0 & \\ 1 & & \end{array}$$

no 1st-column sign changes $\Rightarrow$ D Hurwitz

G proper $\Rightarrow$ BIBO stable

3) An LTI system has eigenvalues 0, -1, and -2. The same system has poles -1 and -2 and zeros 1, 0, and -3. All eigenvalues, poles, and zeros have multiplicity 1.

a) Find the system transfer function.

$$G(s) = \frac{s(s-1)(s+3)}{(s+1)(s+2)} = \frac{s^3+2s^2-3s}{s^2+3s+2}$$

b) Write a differential equation for the system.

$$\Delta(s) = s(s+1)(s+2) = s^3+3s^2+2s$$

$$G(s) = \frac{s^2(s-1)(s+3)}{\Delta(s)} = \frac{s^4+2s^3-3s^2}{s^3+3s^2+2s}$$

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = \ddddot{u} + 2\dddot{u} - 3\ddot{u}$$

c) Does the system have a hidden mode?

0 is an eigenvalue, but not a pole.
Therefore, 0 is a hidden mode.

d) Is the system asymptotically stable?

eigenvalue 0 ⇒ not asymptotically stable

e) Is the system BIBO stable?

G improper ⇒ not BIBO stable

1) Consider the system governed by the differential equation

$$y^{(4)} - 2\ddot{y} + y = \dddot{u} - 2\ddot{u} + \dot{u}.$$

a) Find the poles of the system.

$$G(s) = \frac{N(s)}{D(s)} = \frac{s^3 - 2s^2 + s}{s^4 - 2s^2 + 1}$$

We need the GCD of N and D.

Divide D by N:

$$\begin{array}{c|ccccc} \underline{2 \quad -1 \quad 0} & 1 & 0 & -2 & 0 & 1 \\ & & & & 0 & 0 \\ & & & -1 & -2 & \\ & & 2 & 4 & & \\ \hline & 1 & 2 & 1 & -2 & 1 \end{array}$$

$Q_1 = s + 2$

$R_1 = s^2 - 2s + 1$

Divide N by R_1:

$$\begin{array}{c|cccc} \underline{2 \quad -1} & 1 & -2 & 1 & 0 \\ & & & -1 & 0 \\ & & 2 & 0 & \\ \hline & 1 & 0 & 0 & 0 \end{array}$$

$Q_2 = s$

$R_2 = 0$

$GCD = R_1 = s^2 - 2s + 1$

Divide D by GCD:

$$\begin{array}{c|ccccc} \underline{2 \quad -1} & 1 & 0 & -2 & 0 & 1 \\ & & & -1 & -2 & -1 \\ & & 2 & 4 & 2 & \\ \hline & 1 & 2 & 1 & 0 & 0 \end{array}$$

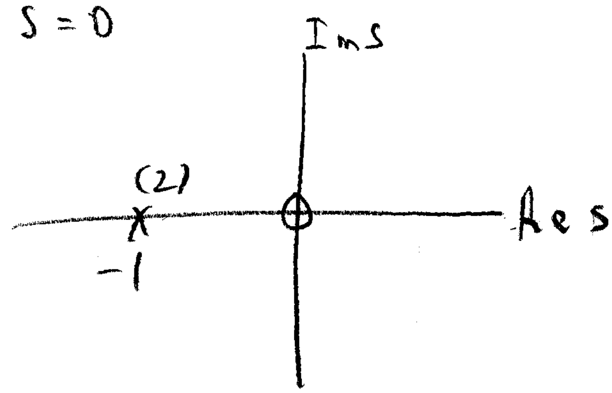
b) Find the zeros of the system.

$$\frac{N}{GCD} = \frac{N}{R_1} = Q_2 = s$$

$\Rightarrow$ single zero at $s = 0$

$$\frac{D}{GCD} = s^2 + 2s + 1 = (s+1)^2$$

$\Rightarrow$ repeated pole at $s = -1$

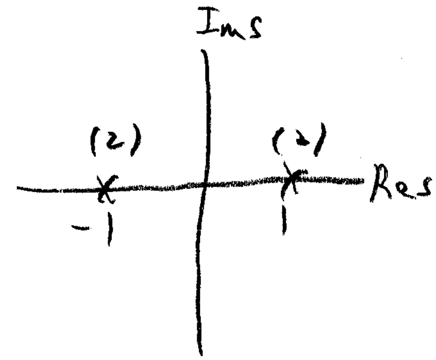


c) Find the eigenvalues of the system.

characteristic polynomial:

$\Delta(s) = s^4 - 2s^2 + 1 = (s-1)^2(s+1)^2$

repeated eigenvalues at $s = \pm 1$



d) Find the hidden modes of the system.

The eigenvalue at $s = 1$ cancels, so it is a hidden mode.

e) Is the system asymptotically stable?

RHP eigenvalue $\Rightarrow$ not asymptotically stable

f) Is the system BIBO stable?

G proper and all poles in open LHP $\Rightarrow$ BIBO stable

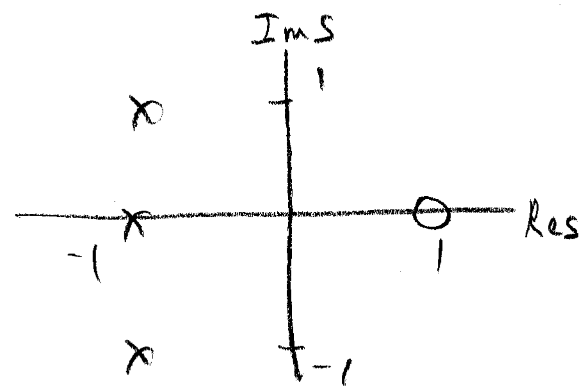
2) Consider the transfer function

$$G(s) = \frac{s-1}{(s+1)(s^2+2s+2)}.$$

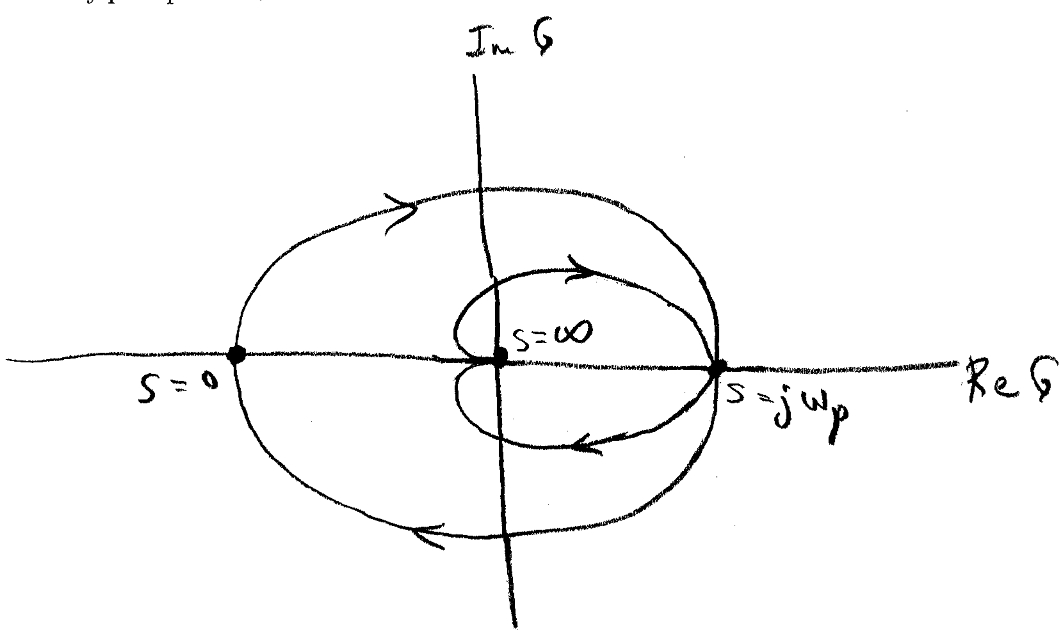
a) Find the poles and zeros of G.

zeros: 1

poles: $-1, -1 \pm j$



b) Draw the Nyquist plot of G.



c) Find the real axis crossings of the Nyquist plot.

$S=0: \quad G(0) = -\frac{1}{2}$

$$s = j\omega_p: \quad \angle G(j\omega) = \angle \frac{j\omega - 1}{j\omega + 1} - \angle\left(-\omega^2 + j2\omega + 2\right)$$

$$= \angle (j\omega - 1)(-j\omega + 1) - \angle\left(j2\omega + 2 - \omega^2\right)$$

$$= \angle\left(j2\omega + \omega^2 + 1\right) - \angle\left(j2\omega + 2 - \omega^2\right)$$

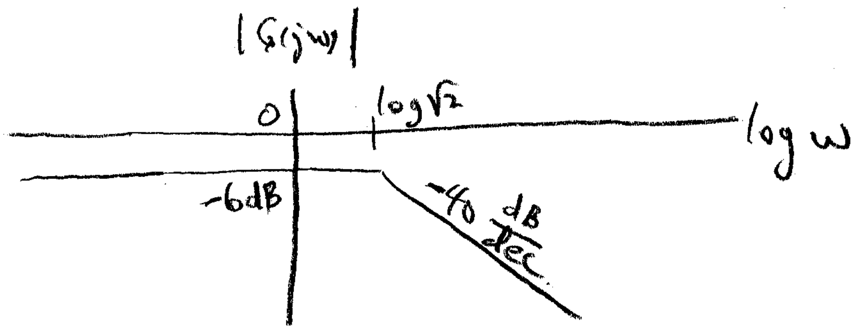
$$= 0$$

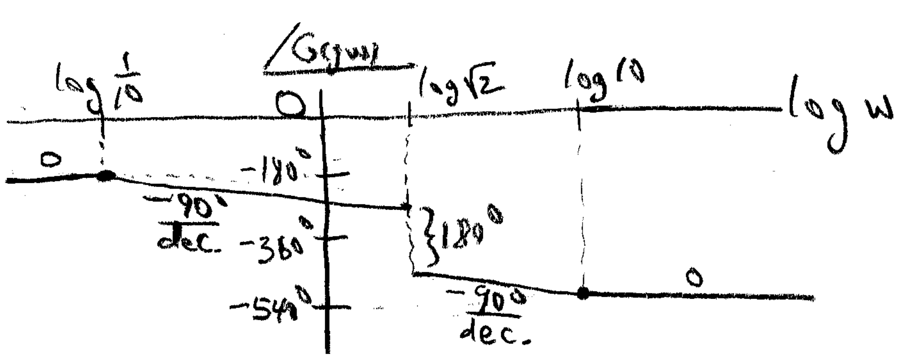
$$\omega^2 - 1 = 2\omega^2 \Rightarrow \omega_p = \sqrt{\frac{3}{2}}$$

$$G(j\omega_p) = \frac{1}{\left|-\omega_p^2 + j2\omega_p + 2\right|} = \frac{1}{\sqrt{\left(2-\frac{3}{2}\right)^2 + 4\left(\frac{3}{2}\right)}} = \frac{2}{5}$$

d) Draw the straight-line approximations for the magnitude and phase Bode plots of G.

$$G(s) = -\frac{1}{2}\,\frac{\frac{s}{-1} + 1}{(s+1)\left(\frac{s^2}{2} + s + 1\right)}$$





3) Determine whether the transfer function

$$G(s) = \frac{1}{s^7 + 3s^6 + 6s^5 + 8s^4 + 8s^3 + 6s^2 + 3s + 1}$$

is BIBO stable.

The poles of G are just the roots of the denominator.

Routh table:

$$\begin{array}{cccc}
1 & 6 & 8 & 3 \\
3 & 8 & 6 & 1 \\
\frac{10}{3} & 6 & \frac{8}{3} & 0 \\
\frac{13}{5} & \frac{18}{5} & 1 & \\
\frac{18}{13} & \frac{18}{13} & 0 & \\
1 & 1 & & \\
0 & & &
\end{array}$$

$\leftarrow$ The denominator is not Hurwitz, so $G(s)$ is not BIBO stable.

3) Design a second-order system

$$H(s) = \frac{A}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

to meet the following specifications:

a) $e_{ss0} = 0$
b) $e_{ss1} \leq .008\ s$
c) $\omega_b \geq 150\ rad/s$
d) $M_r \leq 1.4$

$$e_{ss0} = 0 \iff A = \omega_n^2$$

$$M_r = \frac{1}{2\xi\sqrt{1-\xi^2}} \iff \xi^4 - \xi^2 + \frac{1}{4M_r^2} = 0$$

$$M_r = 1.4 \iff \xi^2 = \frac{1 \pm \sqrt{1-\frac{1}{M_r^2}}}{2} = .15, .85$$

$$.15 \leq \xi^2 \leq .85 \iff .39 \leq \xi \leq .92$$

For $\xi \geq \frac{1}{\sqrt{2}}$, $M_r = 1$, so we only need $\xi \geq .39$.

Let $\xi = .39$.

$$e_{ss1} = 2\frac{\xi}{\omega_n} \leq .008 \iff \omega_n \geq \frac{2\xi}{.008} = 96.8\ \frac{rad}{s}$$

$$\omega_b = \omega_n\sqrt{1-2\xi^2+2\sqrt{1-\xi^2+\xi^4}} \geq 150$$

$$\omega_n \geq \frac{150}{\sqrt{1-2\xi^2+2\sqrt{1-\xi^2+\xi^4}}} = 93.6\ \frac{rad}{s}$$

Let $\omega_n = 96.8\ \frac{rad}{s}$.

$$A = \omega_n^2 = 9380$$

$$H(s) = \frac{9380}{s^2 + 75s + 9380}$$