ECE334 HOMEWORK 5

0 [reading] text sections 1.1.5, 2.2.7, 7.1.1, 7.1.2, 7.1.3.

1 [complex eigenvectors]

Consider the differential equation $\dot{x} = Ax$, x in $\mathbb{R}^3$ where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -3 \\ 1 & 3 & 2 \end{pmatrix}$

First think of x as a complex vector x in $\mathbb{C}^3$ and diagonalize A (that is, find its matrix relative to a basis of eigenvectors) and find the transformation P which diagonalizes A (that is, find P such that $\Lambda = P^{-1}AP$ is diagonal.)
Now think of x as real and find a good basis of real vectors for A and the matrix of A with respect to this basis as indicated in the lectures; that is, use a good basis for conjugate complex pairs of eigenvectors to arrive at a matrix in the form $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & -\omega \\ 0 & \omega & \alpha \end{pmatrix}$
Now solve the differential equation and sketch the behaviour of the solutions in $\mathbb{R}^3$ using the good basis.

2 Consider 3 identical beads of mass 1 sliding on a straight, frictionless horizontal wire. The beads are joined into one assembly by 2 equal springs of spring constant 1. The beads and springs are in the order: bead 1, spring, bead 2, spring, bead 3 along the wire. The system state is $(x_1, x_2, x_3, v_1, v_2, v_3)$ where x_i and v_i are the position and velocity of bead i along the the wire. Find the system modes (eigenvectors) and their natural frequencies (related to the eigenvalues). Obtain the decoupled system equations in the coordinates relative to the eigenvectors. Interpret the eigenvectors and eigenvalues physically, that is, give a physical description of the system modes. The use of Matlab or another package to find the eigenvectors is recommended (If you are allergic to computers, I can give hints to simplify the hand calculations). If you find this problem too hard to tackle all at once, first do the problem for one bead, and then for two beads.

3 [circles in funny norms]
(a) Write down the definition of a norm $|\ |$ on a vector space V.
(b) Prove that $|x| = \text{Max}\{|x_1|, |x_2|\}$ is a norm on $\mathbb{R}^2$. [Note that the notation $|x_1|$ means the absolute value of the real number x_1 (absolute value is a norm on $\mathbb{R}$) whereas the notation $|x|$ means the norm of the vector x.] Sketch the "circle" with respect to this norm. Here we define a "circle" to be the set of vectors x such that $|x| = 1$.
(c) Sketch the "circle" with respect to the norm $|x| = |x_1| + |x_2|$.
(d) Suppose that two locations in downtown Manhattan are determined by a vector in $\mathbb{R}^2$ with the first coordinate measured along the streets (East-West) and the second coordinate measured along the avenues (North-South). What norm is relevant to the task of walking between the two locations? (You can do this question for driving between the two locations if you must, but New York City is easier and more fun on foot.)
(e) You have invested in a factory which produces two products and you plot the production for each day in $\mathbb{R}^2$ with units of dollars on both axes. Assuming that your primary interest is the total sales revenue, what norm is appropriate to measure the change between production on two successive days?

4 [uniform norms on matrices] Prove starting from the definition of the uniform norm $||\ \ ||$ that
(a) $|Av| \le ||A||\,|v|$ for all $v \in \mathbb{R}^n$.
(b) $||AB|| \le ||A||\,||B||$
(c) Find the uniform norm of $\begin{pmatrix} 3 & 2 \\ 0 & 2 \end{pmatrix}$. (Assume the Euclidean norm $|x| = \sqrt{x_1^2 + x_2^2}$ on $\mathbb{R}^2$.)

5 [describing a dynamical system] Consider two wheeled carts of mass M which are joined by a spring of spring constant K. The carts move in one line on level ground and are at positions z_1 and z_2 respectively. One pair of wheels on each cart is powered by a DC electric motor of inertia J, torque constant k and resistance R. The powered wheels on cart 1 are at angle θ_1 and the powered wheels on cart 2 are at angle θ_2. r is the radius of the cart wheels and the linear and angular motions are constrained by $z_i = r\theta_i$, $\dot{z}_i = r\dot{\theta}_i$, $\ddot{z}_i = r\ddot{\theta}_i$ for $i = 1, 2$ (that is, the wheels do not slip). r is also the ratio of motor torque to the linear force applied to the cart. τ_1 and τ_2 are the motor load torques. The system inputs are the voltages e_1 and e_2 on the motors. Convince yourself that the system is described by the following equations:

$$
\begin{aligned}
M\ddot{z}_1 &= K(z_2 - z_1) - \tau_1/r \\
J\ddot{\theta}_1 &= -k^2\dot{\theta}_1/R + ke_1/R + \tau_1 \\
M\ddot{z}_2 &= K(z_1 - z_2) - \tau_2/r \\
J\ddot{\theta}_2 &= -k^2\dot{\theta}_2/R + ke_2/R + \tau_2
\end{aligned}
$$

(a) Obtain the differential equations of the system in state space form $\dot{x} = Ax + Bu$ using the state $x = (z_1, z_2, \dot{z}_1, \dot{z}_2)^t$ and input $u = (e_1, e_2)^t$. (The motor load torques τ_1 and τ_2 need to be eliminated from the equations. When you have obtained the differential equations, it is convenient to write $\alpha = (JR + MRr^2)^{-1}$.)
(b) Draw a system block diagram (the text calls this a simulation diagram).

The following parts use Matlab or Mathematica and the system data:
$M = 1$ kg, $K = 40$ N/m, $k = 2$ Vs, $R = 100\ \Omega$, $r = 2$ cm, $J = 1$ kg m^2.

(c) Find the characteristic equation and the eigenvalues.
(d) Let the outputs be the cart positions $y = (z_1, z_2)^t$.
Write down the C matrix of the output equation $y = Cx$.
Compute $(sI - A)^{-1}$.
Compute the 2×2 matrix of transfer functions from the inputs to the outputs $G(s) = C(sI - A)^{-1}B$.