ECE717 HOMEWORK 2

1 [linear independence and bases] Prove that in a vector space of dimension n, every set $\{x_1, x_2, ..., x_n\}$ of n linearly independent vectors is a basis. [Hint, suppose that y is any vector, then $\{x_1, x_2, ..., x_n, y\}$ must be linearly dependent]

2 [change of basis] Let α be a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ and $\{e_1, e_2, e_3\}$, $\{f_1, f_2, f_3\}$ be bases for $\mathbb{R}^3$. $\{f_1, f_2, f_3\}$ is related to $\{e_1, e_2, e_3\}$ by $f_j = \sum_k P_{jk} e_k$, $j = 1, 2, 3$ where P is an invertible matrix. The matrix A of α with respect to the basis $\{e_1, e_2, e_3\}$ is given by $\alpha(e_j) = \sum_i A_{ij} e_i$, $j = 1, 2, 3$. Derive the formula for the matrix of α with respect to the basis $\{f_1, f_2, f_3\}$.

3 [direct sum and bases] Let V and W be vector spaces with corresponding bases $\{v_1, v_2, \ldots v_j\}$ and $\{w_1, w_2, \ldots w_k\}$. Prove that $\{v_1, v_2, \ldots v_j, w_1, w_2, \ldots w_k\}$ is a basis of $V \oplus W$ and deduce that dimension($V \oplus W$)=dimension(V)+dimension(W). Give an example to show that it is not generally true that dimension($V + W$)=dimension(V)+dimension(W).

4 [kernels and ranges] Let α be a linear map from $\mathbb{R}^3 \to \mathbb{R}^2$. Write down the definitions of kerα and rangeα. Prove that kerα is a subspace of $\mathbb{R}^3$ and that rangeα is a subspace of $\mathbb{R}^2$. Give an example of α with a 2 dimensional kernel and a one dimensional range.

5 [a real life example of kernels and ranges] Let

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

What is the rank of A? Find a basis of the kernel K and range R of A. Draw pictures of K and R in $\mathbb{R}^3$. Under what conditions does $Ax = b$ have a solution for x and is the solution unique?

6 [evalues and evectors] (a) Write down the definition of an eigenvalue and eigenvector.
(b) Prove that a real symmetric matrix A has all eigenvalues real. (Hint: consider wAv^* where right and left eigenvectors v and w correspond to the eigenvalue λ.)
(c) Give an example of a 2×2 real matrix with complex eigenvalues.

7 [Using evalues and evectors to figure out linear systems] Compute the eigenvalues and eigenvectors of the following matrices. Find the matrix of the corresponding linear map relative to a basis of eigenvectors. Sketch the solution trajectories in $\mathbb{R}^2$ of $\dot{x} = Ax$ for each case indicating the invariant subspaces associated with the eigenvectors. Sketch each case in the original coordinates and (except for case (b)) relative to the basis of eigenvectors.

(a) [sink] $\begin{pmatrix} -3 & 4 \\ -0.75 & -7 \end{pmatrix}$

(b) [spiral sink] $\begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix}$

(c) [saddle] $\begin{pmatrix} -6 & -4 \\ 4 & 11 \end{pmatrix}$

(d) [non-trivial Jordan block] Do as much of this question as you can in the case $\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$.
What goes wrong here?

8 Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 0 & 0 & 0 & -24 \\ 1 & 0 & 0 & 50 \\ 0 & 1 & 0 & -35 \\ 0 & 0 & 1 & 10 \end{pmatrix}$ and the matrix of the corresponding linear map relative to a basis of eigenvectors. Write down a matrix whose characteristic polynomial is $\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$.