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THE IRRELEVANCE OF LOAD DYNAMICS FOR THE LOADING MARGIN TO VOLTAGE COLLAPSE AND ITS SENSITIVITIES

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Abstract: The loading margin to a saddle-node or fold bifurcation is a useful and basic index of proximity to voltage collapse. Sensitivities of the loading margin can be used to select controls to avoid voltage collapse. This paper justifies the use of static power models to compute loading margins and their sensitivities. In particular, dynamic load models may be reduced to static equations and then simplified without affecting loading margins and their sensitivities. Alternatively, if dynamic models are not well known, the computations may be done with static models and conclusions drawn about a general and sensible class of underlying dynamic models. Explicit assumptions and mathematical underpinnings are presented.

1. Introduction

Dynamic voltage collapse due to loss of a stable operating equilibrium in a saddle-node or fold bifurcation is an important mechanism for power system instability [11]. Therefore it is useful to compute the location of fold bifurcations of power system models and their proximity to current operating conditions. The proximity to fold bifurcation can be measured by computing the loading margin; that is, the increase in loading required to obtain fold bifurcation. If the loading margin is dangerously small, then controls to steer the power system away from the fold bifurcation can be selected by computing the sensitivity of the loading margin to control actions [14]. The sensitivity of the loading margin to any power system parameter is also easy to compute.

Here the terms saddle-node bifurcation and fold bifurcation may be regarded as interchangeable, but a distinction is made between them in Appendix A. In the particular case of the power system loading being expressed in terms of real and reactive load powers, the loading margin is also called the load power margin.

Section 3 of the paper explains and proves the following observation:
Loading margins to fold bifurcations and their sensitivities depend only on the static parts of dynamic power system models.
For example, a dynamic power system model

$$\dot{x} = f(x, \lambda)$$

may be reduced to the static equations

$$0 = f(x, \lambda)$$

without affecting the loading margin to a fold bifurcation. Although voltage collapse due to fold bifurcation is dynamic and requires dynamic power system models to be properly *understood*, the *computation* of loading margin and its sensitivities only requires knowledge of the static parts of the dynamic models. This simplification is of great practical value because the dynamic parts of load models are often poorly known. We can compute loading margins and their sensitivities using the better known and simpler static load models, while assuming a very general form for the poorly known dynamics which underlie the static equations. This observation generalizes more limited results in [13, 14, 5] and supports the reductions in [27] of differential-algebraic power system equations to static models.

Once static power system models $0 = f(x, \lambda)$ are obtained, they can be advantageously simplified by algebraic manipulation. Section 4 describes useful algebraic manipulations which do not affect the loading margins and their sensitivities. Section 5 states the further assumptions needed to conclude that the loading margin to fold bifurcation is the minimum loading margin to *any* instability.

We choose to measure proximity to voltage collapse with a loading margin because it is an accurate and basic index, appreciable by all, which takes full account of system limits and nonlinearities. Observe that every paper on voltage collapse indices not explicitly using loading margin implicitly acknowledges loading margin by using it as the horizontal scale when the performance of the proposed index is graphed.

2. Fold bifurcation and geometry

The power system state vector x typically includes bus voltages and angles as well as other system states. The parameter vector λ includes measures of system loading such as real or reactive power demand at a bus or more general loading parameters for loads with more elaborate load models. Suppose the power system is operating with state x_0 and parameter λ_0. In order to measure the proximity to voltage collapse, the effect of increasing the loading parameters in λ in some given pattern is computed by a continuation method [e.g. 4,8,1,20] or a quasistatic simulation method [27,21] until a fold bifurcation is encountered at (x_*, λ_*). (Recall that continuation solves for a succession of equilibria as the loading is increased and takes account of power system limits as they occur.) The loading margin is then the distance $|\lambda_* - \lambda_0|$.

We make the following assumption

Assumption 1. Near the bifurcation at (x_*, λ_*), the system is assumed to have dynamics specified by parameterized, smooth differential equations $\dot{x} = f(x, \lambda)$.

Assumption 1 allows the voltage collapse to be analyzed and understood and the sensitivities needed to avoid the bifurcation to be computed. In a fold bifurcation, two equilibria coalesce and disappear. If one of the equilibria is stable, then the dynamic consequences of the fold bifurcation are well approximated by movement, slow then fast along a particular dynamic trajectory called the unstable part of the center manifold of x_* [11].

There is also much information available from eigenvectors of the Jacobian $f_x|_*$ about the geometry of the fold bifurcation. The essential information about the fold bifurcation that needs to be determined is its location (x_*, λ_*), the right eigenvector v and the normal vector N. (The right eigenvector v corresponds to the zero eigenvalue of the singular Jacobian $f_x|_*$. The normal vector $N = wf_\lambda|_*$, where w is the left eigenvector corresponding to the zero eigenvalue of $f_x|_*$.) These have the following engineering uses and interpretations:

(a) The critical loading λ_* determines the loading margin $|\lambda_* - \lambda_0|$ which describes the proximity of the current loading λ_0 to voltage collapse.

(b) The right eigenvector v describes both the direction in state space in which the equilibrium moves just before the bifurcation and the initial direction in state space along which the dynamic voltage collapse occurs. In particular, v quantitatively predicts the relative amounts of voltage decline at buses as the bifurcation is approached and as the initial part of the dynamic collapse occurs [11].

(c) N is the normal vector to the set of critical loadings in parameter space at which fold bifurcations occur. N determines the optimum direction to move in parameter space to avoid fold bifurcation. In particular, it determines the combination of loads to shed to optimally move away from bifurcation [12]. More generally and importantly, N can be used to easily compute the first order sensitivity of the loading margin to *any* power system parameters or controls [14,3]. This allows effective controls to be selected to optimally steer the power system to a condition with a larger loading margin to voltage collapse. For example, the effect on the loading margin of adding 1 MVAR reactive power support at any system bus can be easily computed. Thus the buses at which reactive power support is most effective can be selected. The computation of the sensitivity of the loading margin also allows the critical and noncritical power system parameters to be identified in terms of their quantitative effect on the loading margin.

3. Reduction to static equations

Since the location of the bifurcation (x_*, λ_*) and the useful quantities v and N are computed from the dynamic equations $\dot{x} = f(x, \lambda)$, one might expect that details of the dynamics of f would affect the computations. However, we explain below and prove in appendix A that this is not the case: (x_*, λ_*), v and N only depend on the static equations $0 = f(x, \lambda)$. This not only simplifies the required modeling but is of great practical value because some power system dynamic equations, particularly load power dynamics, are poorly known. However, we retain the notion that there are some dynamics underlying the static equations so that we can understand the voltage collapse. We need only assume a very general form for these dynamics to compute (x_*, λ_*), v and N for these dynamic equations from the corresponding static equations.

First we explain these ideas in a simple case of a single differential equation f in one state variable x and one parameter λ:

$$\dot{x} = f(x, \lambda) = -x^2 - \lambda + 1$$

The set Z is the nose curve parabola (or bifurcation diagram) $\lambda = -x^2 + 1$

$$Z = \{(x, \lambda) \mid 0 = -x^2 - \lambda + 1\}$$

and the dynamics for each fixed value of the parameter λ are indicated by the arrows in Figure 1. There is a generic fold bifurcation at the tip of the nose curve $(x, \lambda) = (0, 1)$. The most important of the conditions confirming this to be a fold bifurcation is that the derivative $f_x = -2x$ vanishes at $(x, \lambda) = (0, 1)$. That is, $f_x|_{(0,1)} = 0$. Since this condition involves f, it would seem to depend on the dynamical equations $\dot{x} = -x^2 - \lambda + 1$. However, an equivalent condition is that Z has a tangent parallel to the state space (that is, vertical in Figure 1) at the bifurcation. This shows that Z has a fold at the bifurcation when Z is viewed along the state space (vertical) axis. Thus the location of the bifurcation only depends on the set Z which is defined by the static equations $-x^2 - \lambda + 1 = 0$. To emphasize this, note that there are many other differential equations with different dynamics but with the same set Z. Examples are:

$$\dot{x} = 2(-x^2 - \lambda + 1)$$
$$\dot{x} = (x^2 + 2)(-x^2 - \lambda + 1)$$

or more generally,

$$\dot{x} = h(-x^2 - \lambda + 1)$$

where h is any smooth function with $h(0) = 0$ and no other zeros and nonzero gradient at zero.

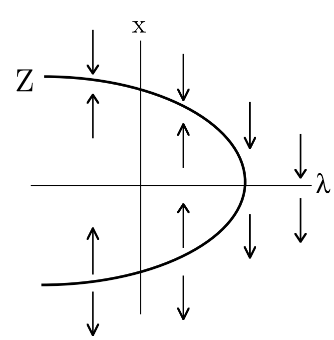

Figure 1. The set Z plus dynamics.

Thus we can compute Z and the location of the bifurcation with any of the static equations $0 = h(-x^2 - \lambda + 1)$ and the dynamics specified by h are irrelevant to this computation. Note that the eigenvalues or gradients f_x of these equations are completely different except that they all vanish at the bifurcation.

Now let the state vector x have n dimensions and the parameter vector λ have m dimensions. The n dimensional state space is denoted by X and the m dimensional parameter space is denoted by Λ. The power system is modeled as n differential equations

$$\dot{x} = f(x, \lambda)$$

Consider the state and parameter values for which f has an equilibrium:

$$Z = \{(x, \lambda) \mid f(x, \lambda) = 0\}$$

The set Z contains hypersurfaces of dimension m in the $m + n$ dimensional combined state and parameter space $X \times \Lambda$. The essence of a continuation method is to compute a curve in Z. Appendix A characterizes the fold bifurcation and its sensitivities in terms of the shape and position of Z. Since the shape and position of Z does not depend on dynamics, the location of the fold bifurcation only depends on the static equations $0 = f(x, \lambda)$ or other static equations yielding the same equilibrium set Z. Moreover, since the vectors v and $\textsc{n}$ can be computed from the geometry of the fold of Z, v and $\textsc{n}$ also do not depend on dynamics and may be computed from static equations. These observations can be stated as follows (for lemma 1 and a more precise version and proof of lemma 2 see appendix A):

Lemma 2. *Suppose* $0 = f(x, \lambda)$ *are static equations with underlying dynamics specified by*

$$\dot{x} = \bar{f}(x) = h\left(f(x, \lambda)\right)$$

where $h(0) = 0$ *and this zero is unique and simple. Then*

(a) f *has a fold bifurcation at* (x_*, λ_*) $\iff$ $\bar{f}$ *has a fold bifurcation at* (x_*, λ_*).

(b) When suitably scaled, $v = \bar{v}$ *and* $\textsc{n} = \bar{\textsc{n}}$.

In the simplest application of lemma 2, h is chosen to be the identity map and the differential equations $\dot{x} = f(x, \lambda)$ are reduced to the static equations $0 = f(x, \lambda)$. If the differential equations contain nonzero time constants $T_1, T_2, \cdots, T_n$ so that $\dot{x} = T^{-1} f(x, \lambda)$, where the matrix $T = \text{diag}\{T_1, T_2, \cdots, T_n\}$, then choosing h in lemma 2 to be multiplication by T^{-1} again yields static equations $0 = f(x, \lambda)$. This shows that the location and sensitivities of fold bifurcations are independent of the value of the time constants. (That this holds for the *location* of fold bifurcations should be obvious upon noting that multiplying the kth row of the Jacobian f_x by T_k multiplies the determinant of f_x by T_k but does not affect where the determinant vanishes. However, the nonzero eigenvalues or singular values of f_x generally depend on the time constants.) One might also want to exploit the freedom to scale the static equations by multiplying them by various constants to improve the conditioning of the Jacobian f_x for numerical computation.

Often static equations are used because the dynamic equations are poorly known. Then lemma 2 gives a very general class of dynamic equations which can be assumed to underlie the static equations. For example, suppose we have power system equations of the form

$$\dot{x}_k = f^k(x, \lambda), \qquad k = 1, \cdots, n_g,$$
$$0 = f^k(x, \lambda), \qquad k = n_g{+}1, \cdots, n.$$

where the algebraic equations express real or reactive power balance at the loads. Consider the load numbered j first and suppose that f^{n-1} and f^n represent real and reactive power balance at load j with state variables voltage magnitude V and angle θ. Suppose that the underlying dynamics at load j are of the form

$$\begin{pmatrix} \dot{V} \\ \dot{\theta} \end{pmatrix} = h^j \left(\begin{pmatrix} f^{n-1}(x, \lambda) \\ f^n(x, \lambda) \end{pmatrix} \right)$$

where $h^j : U \to \mathbb{R}^2$ is a vector valued vector function on U, where $0 \in U \subset \mathbb{R}^2$. We suppose that the Jacobian of h is nonsingular at the origin. (This is reasonable since independent first order variations in real and reactive power balance would be expected to give independent first order variations in $\dot{V}$ and $\dot{\theta}$.) Then if all the loads have similar form to load number j, a function h suitable for lemma 2 can be constructed from the functions similar to h^j for the loads and identity functions for the first n_g equations. Thus lemma 2 shows that for understanding the problem, we can assume general and sensible underlying dynamics for each load, but for purposes of computing fold bifurcations, the static model

$$0 = f^k(x, \lambda), \qquad k = 1, \cdots, n.$$

can be used with no loss of accuracy. This example shows how differential-algebraic power system models can be reduced to static (algebraic) equations for computation of loading margins and their sensitivities. The computations are valid for a general class of dynamics underlying the static equations. For another example of the use of this reduction see [27, section 3.2].

4. Simplifying the algebraic equations

Once static equations are obtained, it is often useful to algebraically manipulate them to simplify them or gain some computational advantage. The following lemma shows that algebraic manipulations which preserve the set Z and the rank of the equations do not affect fold bifurcations (for a more precise and complete version and proof see appendix A):

Lemma 4. (informal version) *Let* $f : X \times \Lambda \to X$ *have* $Z = \{(x, \lambda) \mid f(x, \lambda) = 0\}$. *Let* $\bar{f} : X \times \Lambda \to X$ *have* $\bar{Z} = \{(x, \lambda) \mid \bar{f}(x, \lambda) = 0\}$. *Suppose that* $Z = \bar{Z}$ *and that at* $z_* \in Z$, *rank* $f_z|_* =$ *rank* $\bar{f}_z|_*$ *where* $z = (x, \lambda)$. *(*$f_z|_*$ *is the Jacobian of* f *with respect to* z *evaluated at* z_*.*) Then*

(a) f *has a fold bifurcation at* z_* $\iff$ $\bar{f}$ *has a fold bifurcation at* z_*.

(b) When suitably scaled, $v = \bar{v}$ *and* $\textsc{n} = \bar{\textsc{n}}$.

Sometimes load dynamics are specified in an implicit form and algebraic manipulation is required to show how the dynamics can be removed using lemma 2. For example, consider the following power system dynamic model with a single dynamic load model shown explicitly:

$$\dot{y} = f^1(y, V, \lambda) \tag{4.1}$$
$$0 = g(\dot{P}^d, \dot{V}, P^d, V, \lambda) \tag{4.2}$$
$$P^d = P(y, V) \tag{4.3}$$

Here V is the load voltage, y are the other state variables, P^d is the real power demanded by the load and the function P evaluates the real power supplied to the load by the electrical network. Equation (4.2) is a general form due to Hill; it includes as a special case his exponential recovery model [19]. Equation (4.3) is real power balance at the load and equation (4.1) are the other dynamic equations. Appendix B shows that if (4.1-4.3) specify a well posed differential equations in the state variables (y, V), then lemmas 2 and 4 can be applied to yield static equations

$$0 = f^1(y, V, \lambda) \tag{4.4}$$

$$0 = g(0, 0, P(y, V), V, \lambda) \tag{4.5}$$

Note that equation (4.5) is the static part of the dynamic load model (4.2), since it corresponds to setting $\dot{P}^d = \dot{V} = 0$ in (4.2).

It is often useful to reduce the number of equations by solving some of the equations for some of the variables and substituting the solved variables in the remaining equations. The inverse of this process, in which new variables are defined and more equations are written, is also sometimes useful because it can improve system sparsity. When changing the static equations in this way, we want to be assured that the fold bifurcations and their geometry are essentially unchanged. This is the substance of the following lemma (for a more precise version and proof see appendix A):

Lemma 5. (informal version) *Let $x \in \mathbb{R}^n$ be written as $x = (a, b)$ where $a \in \mathbb{R}^{\bar{n}}$ and $b \in \mathbb{R}^{n-\bar{n}}$. Write the n equations $0 = f(x, \lambda)$ as $0 = \begin{pmatrix} f^1(a, b, \lambda) \\ f^2(a, b, \lambda) \end{pmatrix}$ where f^1 consists of $\bar{n}$ equations and f^2 consists of $n - \bar{n}$ equations. Suppose that $f^2_b|_*$ is invertible so that the equations $0 = f^2(a, b, \lambda)$ are solvable (at least locally) for b so that $b = h(a, \lambda)$. Define the $\bar{n}$ equations $\bar{f}(a, \lambda) = f^1(a, h(a, \lambda), \lambda)$. Then f has a fold bifurcation at $z_* = (a_*, b_*, \lambda_*)$ $\iff$ $\bar{f}$ has a fold bifurcation at $\bar{z}_* = (a_*, \lambda_*)$. Moreover, for suitable scaling of $\bar{n}$ and $\bar{v}$, $N = \bar{N}$ and $v = \begin{pmatrix} \bar{v} \\ h_a|_* \bar{v} \end{pmatrix}$.*

A simple example of the use of lemmas 2 and 5 occurs when the dynamic equations include $\dot{\delta} = \omega$, where δ is the power angle of a generator. Lemma 2 reduces $\dot{\delta} = \omega$ to the algebraic equation $0 = \omega$ and lemma 5 allows ω to be set to zero in all the other power system equations and omitted from the system state.

Lemma 5 also allows equation (4.4-4.5) to be rewritten with an additional state P^d as

$$0 = f^1(y, V, \lambda)$$
$$0 = g(0, 0, P^d, V, \lambda)$$
$$P^d = P(y, V)$$

if desired, without affecting the fold bifurcation.

A more substantial application of lemma 5 can be seen to yield the Liapunov Schmidt reduction (choose a of lemma 5 to be a coordinate along v and define $f^1 = wf$, etc.) [10].

Note that each of the lemmas which reduce dynamic equations to static equations or simplify the static equations states the relation between v and N before and after the reduction or simplification. Therefore, if we compute v and N with the simplified static equations, we can work back through the applicable lemmas to deduce v and N for the dynamic equations. That is, the geometry of the fold of the dynamic equations is calculable from the geometry of the fold of the simplified static equations and vice versa. The most striking instance of this property is that the initial direction v of movement of the dynamic voltage collapse can be determined from static equations!

Cañizares [5] proves by direct calculation that a set of differential-algebraic equations modeling an AC/DC power system can be reduced to static equations while preserving the loading margin and the bifurcation geometry. Lemmas 2 and 5 apply to the AC/DC power system model and provide alternate proofs of Theorems 1 and 2 of [5] (use the versions of lemmas 2 and 5 in Appendix A to obtain Theorem 2).

5. The context of other instabilities

So far we have only considered the computation of loading margin to a fold bifurcation of an equilibrium. The previous assumption 1 suffices if we are only concerned with this narrow focus. There are other possible instabilities and transient dynamics to be considered in determining the loading margin to the *first* instability of the power system as the loading is increased. We state some additional assumptions necessary if we are to conclude that the fold bifurcation is this first bifurcation and hence the applicable instability mechanism for the given loading increase. The reader should refer to [27], especially the second scenario of section 2.2, for a previous account of some of the ideas in this section.

Assumption 2. The system is assumed to governed by smooth differential equations parameterized by λ as the loading increases. However, the differential equations may undergo discrete changes when limits or structural changes are encountered. Limits important to voltage collapse include generator reactive power limits and tap changing transformer limits. A line tripping or a generator outage are examples of structural changes.

Assumption 3. The initial operating condition (x_0, λ_0) is assumed to be asymptotically stable. This is usually known either by assuming a zero loading λ_0 at which the asymptotically stable equilibrium x_0 is known or by previous or current operating experience at a higher loading λ_0. (If the power system operates at (x_0, λ_0) without transients, then (x_0, λ_0) must be stable.)

Then there are three typical ways in which stability of the equilibrium can be lost as the loading is increased from λ_0:

(1) *Fold bifurcation instability.* The stable equilibrium disappears in a fold bifurcation. This instability occurs generically and locating and avoiding this bifurcation is the focus of this paper. Note that a fold bifurcation *must* occur eventually because all solutions disappear for sufficiently high loading.

(2) *Instability due to discrete change.* A stable equilibrium may disappear or become unstable when the differential equations undergo discrete changes. For example, when a generator encounters a reactive power limit, the system equations change in such a way that the equilibrium persists but can possibly change stability [13]. If a line or generator is tripped, then the system equations change in such a way that equilibrium can possibly be lost.

(3) *Hopf bifurcation instability.* The stable equilibrium persists, but becomes unstable in an oscillatory instability associated with Hopf bifurcation. This instability occurs generically. Complete dynamic models are required to detect or analyze a Hopf bifurcation. We do not consider oscillatory instabilities in this paper. There are interesting possible mechanisms for voltage collapse involving Hopf bifurcations [29] but we have not seen convincing operating experience of oscillations playing a significant role in voltage collapse incidents. In general, leaving aside the thrust of this paper, the importance of power system oscillations and the possibilities of more exotic power system dynamical behaviors are strong incentives to develop good dynamic models including dynamic load models.

It is also possible in exceptional cases for combinations of instabilities (1) (2) and (3) to occur simultaneously, but we ignore these nongeneric possibilities.

If the discrete change or Hopf bifurcation instabilities do not occur as the loading is increased, then the equilibrium will always lose stability in a fold bifurcation. However, an additional assumption is needed to ensure that the system state stays near enough to the equilibrium as the loading is increased to guarantee that stability is not lost due to large signal transient effects:

Assumption 4. The system state tracks the system equilibrium sufficiently closely as the parameters change. This assumption is known as the quasistatic assumption [11]. Deviation of the state from the stable equilibrium can occur when the parameter rate of change becomes large enough or when discrete changes to the equations occur. In either case, transients occur and these transients are assumed to restore the state near to the stable equilibrium in such a way that the particular form and size of the transient can be neglected.

The ideas of this section so far can be summarized as follows:
Continuation methods can be applied to compute the fold bifurcation starting from a known stable condition at low loading. If stability is not lost due to oscillatory instabilities or limits or structural changes as the loading is slowly increased, then the stability must eventually be lost in a fold bifurcation and the loading at this fold bifurcation will determine the minimum system loading margin to any instability.

As regards checking whether instability occurs due to oscillatory instabilities (Hopf bifurcations), we suggest that it useful to analyze and avoid oscillatory instabilities separately from the fold bifurcation. Although Hopf bifurcations and fold bifurcations can occur in the same power system models [9,2,24,7,29,15] there is not yet evidence that their instability problems are closely related. For example, one might argue that if excitation systems are well designed, Hopf bifurcation instabilities encountered at higher loadings when studying voltage collapse models would vanish. (The parameter sensitivities for Hopf and fold bifurcations can be studied and compared using the techniques of [15].) From a practical modeling point of view, the Hopf bifurcation requires a complete dynamic model to be known and so is better analyzed separately. (We note that eigenvalue, singular value and Hopf analyses of differential-algebraic equations are only valid if the underlying dynamics enforcing the algebraic constraints are assumed to be both stable and fast.)

We hope to develop analytic tools for checking for instabilities due to limits or structural changes in future work.

Even with the assumptions above, the continuation approach to computing loading margins and their sensitivities has at present significant limitations. For example, the timing and complex interaction of discrete and continuously evolving events are not suitably represented. However, a time domain simulation can represent dynamic models in great detail and readily reproduce complex series of events. The loading margin and sensitivity approach is complementary to time domain simulation. While the loading margin approach must make some restrictive assumptions, it does provide useful sensitivity information and insight not available from time domain simulations.

6. Conclusions

This paper explains and justifies how static power system models can be used to compute the loading margin to voltage collapse due to fold or saddle node bifurcation and its sensitivities to parameters and controls. The static models are understood to have underlying dynamics which are responsible for the dynamic voltage collapse as a consequence of the fold bifurcation.

If power system dynamic equations are given, then we show that reducing the dynamic equations to static equations by setting the derivatives of the state variables to zero does not affect the loading margin or its sensitivities. Once the static equations are obtained, they may be algebraically simplified in several ways, again without affecting the loading margin or its sensitivities. Conclusions about the geometry of the fold bifurcation of the dynamic equations are available from the simplified static equations. This is illustrated by showing how a general class of dynamic load models due to Hill [19] can be reduced to static models. We do not advocate any particular static load models in this paper, but we do point out that it is essential to make an explicit choice of the load parameters to be included in the parameter vector λ when specifying a load model.

More commonly, the complete dynamic equations of the power system are not well known. However, we can compute load power margins and their sensitivities with static power system equations and the paper shows that the results are accurate for a general and sensible class of underlying dynamic equations.

It is a substantial advantage for a voltage collapse index to be independent of dynamics both in simplifying and optimizing the power system model for computations and in allowing the computation to be done accurately despite poorly known dynamics. The paper shows that the load power margin is independent of dynamics and we suspect that this useful property is also shared by the energy function index. (Note that [23] gives an interpretation of an energy function index as the area enclosed by the intersection of Z with a plane spanned by the real power and angle at a bus; this interpretation is independent of the dynamics.)

Indices based on eigenvalues [e.g. 16] or singular values [e.g. 22, 25] depend on the linearization of the system at an operating equilibrium. Therefore these indices require for a power system model either a full set of differential equations or differential-algebraic equations with the assumption or knowledge that the algebraic equations are enforced by underlying dynamics that are both fast and stable. In particular, voltage collapse indices based on eigenvalues or singular values depend on load dynamics.

This paper provides justification and explicit assumptions for computations of loading margins and their sensitivities with static power system models. There are some technical differences between the mathematical definitions and methods appropriate for fold bifurcations of static and dynamic equations. The appendices of the paper contribute definitions and methods appropriate for fold bifurcations of static equations.

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Appendix A: Definitions and lemmas

A definition of a fold bifurcation of equations f is presented and its relation to the standard definition of a saddle node bifurcation is discussed. The fold bifurcation of equations f is characterized in terms of a fold of the set

$$Z = \{(x, \lambda) \mid f(x, \lambda) = 0\}$$

which only depends on static parts of the power system model. Lemmas needed for sections 3 and 4 are stated and proved.

Let the state space X be $\mathbb{R}^n$ and the parameter space Λ be $\mathbb{R}^m$. Dynamic equations are $\dot{x} = f(x, \lambda)$ where $f : U \to TX$ is a smooth set of differential equations and U is an open set of $X \times \Lambda$ containing (x_*, λ_*). (TX is the tangent space of X.) We will also define a fold bifurcation for smooth static equations $0 = f(x, \lambda)$ where $f : U \to X$.

It is useful in this appendix to distinguish between saddle-node bifurcations and fold bifurcations. We make this distinction by defining the bifurcations slightly differently.

Definition: *Let $f : U \to X$ or $f : U \to TX$ be a smooth function and let the dimension of X be n. f has a* **fold bifurcation** *at $(x_*, \lambda_*) \in U$ if it satisfies conditions* F(a)-F(d)*:*

F(a) $f(x_*, \lambda_*) = 0$
F(b) $f_x|_*$ *has rank* $n - 1$
F(c) $wf_\lambda|_* \neq 0$
F(d) $wf_{xx}|_*(v, v) \neq 0$

where v and w are nonzero vectors satisfying
$f_x|_* v = 0$ *and* $wf_x|_* = 0$.

Condition F(a) states that (x_*, λ_*) is a root or equilibrium of f. Condition F(b) states that the Jacobian f_x is singular at the bifurcation and that the kernel $< v >$ of $f_x|_*$ is one dimensional. The transversality conditions F(c) and F(d) help to ensure that the fold bifurcation is not degenerate.

The standard definition for a generic saddle-node bifurcation can be obtained by adding another condition to F(a)-F(d):

Definition: *f has a* **saddle-node bifurcation** *at (x_*, λ_*) if f has a fold bifurcation at (x_*, λ_*) and*
(e) *$f_x|_*$ has a simple zero eigenvalue.*

There are two possibilities for f to satisfy F(b). The generic possibility is that $f_x|_*$ has rank $n-1$ and a unique simple zero eigenvalue. In this case f satisfies the conditions for a generic saddle node bifurcation at (x_*, λ_*) and the dynamics near (x_*, λ_*) are as described in [11]. The exceptional possibility is that $f_x|_*$ has rank $n - 1$ and a nonsimple zero eigenvalue. That is, there is a nontrivial Jordan block with zero eigenvalues of geometric multiplicity one and algebraic multiplicity greater than one. The consequence of the nontrivial Jordan block is that the dynamics of f near (x_*, λ_*) are more complicated. For example, in the case of Jordan block $\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, there are Hopf and saddle connection bifurcations nearby (see [18, section 7.3]). This bifurcation is codimension 2 but can occur in power system models (see point A in [28]).

However, the conditions F(a)-F(d) are sufficient to guarantee that near λ_* there is locally a critical hypersurface in parameter space at which two equilibria coalesce. On one side of the hypersurface near λ_* there are two equilibria and on the other side near λ_* there are none. This is proved in [10, section 6.2]. Those skeptical about this fact in the exceptional case should look at Figure 7.3.1 of [18].

One theme of this appendix is that if we are concerned with algebraic equations and their equilibria, then F(a)-F(d) are the right conditions for fold bifurcation. This is the context of singularity theory and in this context no distinction is made between the dynamics at a generic saddle node and the dynamics at a fold bifurcation with a nonsimple eigenvalue, because these phenomena are indistinguishable by examining the equilibria only. Indeed it can be inappropriate in this context to discuss eigenvalues of algebraic equations because changes to the equations which preserve their equilibria generally alter the eigenvalues. For example, multiplying one of the algebraic equations by two does not affect the equilibria but generally does alter the eigenvalues.

The fold bifurcation defined by F(a)-F(d) is a generic codimension 1 event in the singularity theory context. In the dynamic vector context, we must additionally require simplicity of the zero eigenvalue in order to obtain a generic saddle node bifurcation and exclude the exceptional codimension 2 event of a nonsimple eigenvalue.

Now we define a fold of a set $Z \subset U \subset X \times \Lambda$. The natural projection $X \times \Lambda \to \Lambda$ is given by $(x, \lambda) \mapsto \lambda$ and we write $\pi : Z \to \Lambda$ for the restriction to Z of the natural projection $X \times \Lambda \to \Lambda$.

Definition: *Let Z be a subset of U and $z_* \in Z$. Z has a* **fold** *at $z_* \in U$ if it satisfies conditions* ZF(a)-ZF(c)*:*

ZF(a) *Z is a submanifold of U of dimension m*
ZF(b) *$\pi_z|_*$ has rank $m - 1$*
ZF(c) $\textsc{n}\pi_{zz}|_*(\textsc{v}, \textsc{v}) \neq 0$

where $\textsc{v}$ and $\textsc{n}$ are nonzero vectors satisfying
$\pi_z|_* \textsc{v} = 0$ *and* $\textsc{n}\pi_z|_* = 0$.

ZF(b) is equivalent to the set Z having a tangent parallel to the state space or having a fold when viewed along this tangent. This tangent direction is given by the right eigenvector $\textsc{v}$. The hypersurface of critical parameters at which there is a fold bifurcation is the projection of the fold in Z onto the parameter space Λ and the normal vector to this hypersurface is $\textsc{n}$.

Lemma 1 is the key lemma which proves that fold bifurcations of f correspond to folds of Z:

Lemma 1. *Suppose $Z = f^{-1}(0)$. Then*

(a) *f has a fold bifurcation at z_* $\iff$
Z has a fold at z_* and rank $f_z|_* = n$.*

(b) *Suppose conditions* F(a)-F(d) *and* ZF(a)-ZF(c) *are satisfied. Choose the scalings of v, w, $\textsc{v}$, $\textsc{n}$ such that $|v| = |\textsc{v}|$ and $|wf_\lambda|_*| = |\textsc{n}|$. Then $\textsc{v} = (v, 0)$, $\textsc{n} = wf_\lambda|_*$ and $\textsc{n}\pi_{zz}|_*(\textsc{v}, \textsc{v}) = -wf_{xx}|_*(v, v)$.*

Proof (a)$\Rightarrow$: Since $f_x|_*$ has rank $n - 1$ and $wf_\lambda|_* \neq 0$, $f_z|_* = (f_x, f_\lambda)\,|_*$ has rank n. Then there is an subset

$U_1 \ni z_*$ open in $X \times \Lambda$ on which f has rank n. Then $Z = (f|_{U_1})^{-1}(0)$ is a submanifold of U of dimension m and ZF(a) is proven.

Since $(f_x|_*, f_\lambda|_*)\binom{v}{0} = 0$ and the only x for which $(f_x|_*, f_\lambda|_*)\binom{x}{0} = 0$ are multiples of v, $TZ|_* \cap TX|_* = \langle(v,0)\rangle$. Since the natural projection $X \times \Lambda \to \Lambda$ has kernel TX, its restriction to Z, π has $\ker \pi_z = TZ \cap TX$. Hence $\ker \pi_z|_* = TZ|_* \cap TX|_* = \langle(v,0)\rangle$ and $\pi_z|_*$ has rank $m-1$ and ZF(b) is proven. Write $\textsc{v} = (v,0)$ and note that $\pi_z|_* \textsc{v} = 0$ and $\textsc{v} \neq 0$. Also write $\textsc{n} = w f_\lambda|_*$ and note that $\textsc{n} \neq 0$.

The natural projection $X \times \Lambda \to X$ is given by $(x,\lambda) \mapsto x$. Write $\chi : Z \to \Lambda$ for the restriction to Z of the natural projection $X \times \Lambda \to X$. Note that $\chi_z \textsc{v} = \chi_z(v,0) = v$. For any $z = (x,\lambda) \in Z$,

$$0 = f(\chi(z), \pi(z))$$

Differentiate to obtain

$$0 = f_x \chi_z + f_\lambda \pi_z \tag{A1}$$

Multiply by w on the left and evaluate at z_* to obtain

$$0 = w f_\lambda|_* \pi_z|_* = \textsc{n} \pi_z|_* \tag{A2}$$

Differentiate (A1) with respect to Z to obtain

$$0 = f_{xx}\chi_z\chi_z + f_x\chi_{zz} + 2f_{x\lambda}\chi_z\pi_z + f_\lambda\pi_{zz} + f_{\lambda\lambda}\pi_z\pi_z$$

Multiply on the left by w and "on the right" by $(\textsc{v},\textsc{v})$, evaluate at z_* and rearrange to obtain

$$\textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) = -w f_{xx}|_*(v,v) \tag{A3}$$

Hence $\textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) \neq 0$ and ZF(c) is proven.

(a)$\Leftarrow$: $f(z_*) = 0$ and F(a) follow immediately from $z_* \in Z = f^{-1}(0)$. Since $\langle \textsc{v} \rangle = \ker \pi_z|_* = TZ|_* \cap TX|_*$, $\textsc{v} \in TX|_*$ and can be written as $\textsc{v} = (v,0)$ where $v \neq 0$. $\textsc{v} \in TZ|_*$ implies that $(f_x|_*, f_\lambda|_*)\binom{v}{0} = f_x|_* v = 0$ and $v \in \ker f_x|_*$. Moreover, if $v_1 \in \ker f_x|_*$, then $0 = f_x|_* v_1 = (f_x|_*, f_\lambda|_*)\binom{v_1}{0}$ and $\binom{v_1}{0} \in TZ|_* \cap TX|_* = \ker \pi_z|_*$. Since $\ker \pi_z|_* = \langle \textsc{v} \rangle = \langle(v,0)\rangle$, v_1 must be a multiple of v. Hence $\ker f_x|_* = \langle v \rangle$ and rank $f_x|_* = n-1$ and F(b) is proven.

Choose $w \neq 0$ such that $w f_x|_* = 0$. $f_z|_* = (f_x|_*, f_\lambda|_*)$ rank n and $f_x|_*$ rank $n-1$ imply that $w f_\lambda|_* \neq 0$ and F(c) is proven. Obtain (A2) as before and use the single dimension of the kernel of $\pi_z|_*$ to deduce that w can be rescaled so that $\textsc{n} = w f_\lambda|_*$. Obtain (A3) as before and deduce that $w f_{xx}|_*(v,v) = -\textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) \neq 0$ and F(d) is proven.

(b): Part (b) follows from the proofs of part (a). □

Lemma 2. *Suppose $0 = f(x,\lambda)$ are static equations with underlying dynamics specified by*

$$\dot{x} = \bar{f}(x) = h(f(x,\lambda))$$

where $h(0) = 0$ and the zero of h is unique and simple. That is, $h(x) = 0 \iff x = 0$ and $h_x|_0$ is nonsingular. Then

(a) *f has a fold bifurcation at (x_*, λ_*) $\iff$ $\bar{f}$ has a fold bifurcation at (x_*, λ_*).*

(b) *Choose the scalings of v, $\bar{v}$, w, $\bar{w}$ such that $|v| = |\bar{v}|$ and $|w f_\lambda|_*| = |\bar{w}\bar{f}_\lambda|_*|$. Then $v = \bar{v}$, $\textsc{n} = w f_\lambda|_* = \bar{w}\bar{f}_\lambda|_* = \bar{n}$ and $w f_{xx}|_*(v,v) = \bar{w}\bar{f}_{xx}|_*(\bar{v},\bar{v})$.*

Proof (a): The uniqueness of the zero of h implies that $0 = \bar{f}(x) = h(f(x)) \iff 0 = f(x)$ and $Z = f^{-1}(0) = \bar{f}^{-1}(0) = \bar{Z}$. Since $\bar{f}_x|_* = h_x|_0 f_x|_*$ and $h_x|_0$ is invertible, rank $f_z|_* = n \iff$ rank $\bar{f}_z|_* = n$. Now Lemma 1(a) yields f has a fold bifurcation $\iff$ Z has a fold and rank $f_z|_* = n \iff \bar{Z}$ has a fold and rank $\bar{f}_z|_* = n \iff \bar{f}$ has a fold bifurcation.

(b): Lemma 1(b) yields (b). □

The essentials of a fold in π are not changed when Z is changed by a diffeomorphism affecting only the state space only as proved in lemma 3. Note that lemma 3 applies when a new state space of different dimension is used.

Lemma 3. *Let Z be a submanifold of $X \times \Lambda$ of dimension m and $\pi : Z \to \Lambda$ be the restriction to Z of the natural projection $X \times \Lambda \to \Lambda$. Suppose π has a fold at $z_* \in Z$ so that $\pi_z|_*$ has rank $m-1$ and $\textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) \neq 0$ where $\textsc{v}$ and $\textsc{n}$ are nonzero vectors satisfying $\pi_z|_*\textsc{v} = 0$ and $\textsc{n}\pi_z|_* = 0$. Let $\bar{Z}$ be a submanifold of $\bar{X} \times \Lambda$ and $\bar{\pi} : \bar{Z} \to \Lambda$ be the restriction to $\bar{Z}$ of the natural projection $\bar{X} \times \Lambda \to \Lambda$. Suppose that there is a diffeomorphism $\phi : Z \to \bar{Z}$ which is onto and which preserves Λ so that $\bar{\pi}\phi = \pi$. Let $\bar{\textsc{v}} = \phi_z|_*(\textsc{v})$ and $\bar{\textsc{n}} = \textsc{n}$ and $\bar{z}_{\bar{*}} = \phi(z_*)$. Then $\bar{\pi}_{\bar{z}}|_{\bar{*}}$ has rank $m-1$, $\bar{\pi}_{\bar{z}}|_{\bar{*}}\bar{\textsc{v}} = 0$, $\bar{\textsc{n}}\bar{\pi}_{\bar{z}}|_{\bar{*}} = 0$, $\bar{\textsc{n}}\bar{\pi}_{\bar{z}\bar{z}}|_{\bar{*}}(\bar{\textsc{v}},\bar{\textsc{v}}) = \textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v})$, and $\bar{\pi}$ has a fold at $\bar{z}_{\bar{*}}$.*

Proof: ϕ a diffeomorphism and $\bar{\pi}\phi = \pi$ implies $\bar{\pi} = \pi\phi^{-1}$ and

$$\bar{\pi}_{\bar{z}} = \pi_z(\phi_z)^{-1} \tag{A4}$$

Hence range $\bar{\pi}_{\bar{z}}|_{\phi(z_1)} =$ range $\pi_z|_{z_1}$. In particular, rank $\bar{\pi}_{\bar{z}}|_{\bar{*}} =$ rank $\pi_z|_* = m-1$. Also

$$\bar{\pi}_{\bar{z}}|_{\bar{*}}\bar{\textsc{v}} = \pi_z|_*(\phi_z|_*)^{-1}\phi_z|_*\textsc{v} = \pi_z|_*\textsc{v} = 0$$

$$\bar{\textsc{n}}\bar{\pi}_{\bar{z}}|_{\bar{*}} = \textsc{n}\pi_z|_*(\phi_z|_*)^{-1} = 0$$

Rearranging (A4) and differentiating yields

$$\bar{\pi}_{\bar{z}\bar{z}}\phi_z\phi_z + \bar{\pi}_{\bar{z}}\phi_{zz} = \pi_{zz}$$

Multiplying by $\textsc{n} = \bar{\textsc{n}}$ and $\textsc{v}$ and evaluating at z_* gives

$$\bar{\textsc{n}}\bar{\pi}_{\bar{z}\bar{z}}|_{\bar{*}}(\bar{\textsc{v}},\bar{\textsc{v}}) = \textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) \tag{A5}$$

The results obtained imply that $\bar{\pi}$ has a fold at $\bar{z}_*$. □

Note that equation (A5) only holds when the scaling of $\bar{v}$ and $\bar{v}$ is such that $\bar{\textsc{v}} = \phi(\textsc{v})$. However, (A5) is usually used to ensure that $\textsc{n}\pi_{zz}|_*(\textsc{v},\textsc{v}) \neq 0 \iff \bar{\textsc{n}}\bar{\pi}_{\bar{z}\bar{z}}|_{\bar{*}}(\bar{\textsc{v}},\bar{\textsc{v}}) \neq 0$ and in this case the scalings of $\bar{\textsc{v}}$ and $\textsc{v}$ are irrelevant.

Lemma 4. *Let $f : X \times \Lambda \to X$ be smooth and have zero set Z. Let $\bar{f} : X \times \Lambda \to X$ be smooth and have zero set $\bar{Z}$. Suppose that $Z = \bar{Z}$ and that at $z_* \in Z$, rank $f_z|_* =$ rank $\bar{f}_z|_*$ where $z = (x,\lambda)$. Then*

(a) *f has a fold bifurcation at z_* $\iff$ $\bar{f}$ has a fold bifurcation at z_*.*

(b) *Choose the scalings of $\bar{v}$, $\bar{w}$, $\bar{\textsc{n}}$ such that $|v| = |\bar{v}|$ and $|w f_\lambda|_*| = |w\bar{f}_\lambda|_*|$. Then $v = \bar{v}$, $\textsc{n} = w f_\lambda|_* = w\bar{f}_\lambda|_* = \bar{\textsc{n}}$ and $w f_{xx}|_*(v,v) = w\bar{f}_{xx}|_*(v,v)$.*

Proof: Follows from lemmas 1 and 3 with ϕ as the identity map. □

Lemma 5. *Let* $f(x,\lambda) = \begin{pmatrix} f^1(a,b,\lambda) \\ f^2(a,b,\lambda) \end{pmatrix}$ *be a smooth function with component functions* f^1, f^2 *where*

$$f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$
$$f^1 : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{n-\bar{n}} \times \mathbb{R}^m \to \mathbb{R}^{\bar{n}}$$
$$f^2 : \mathbb{R}^{\bar{n}} \times \mathbb{R}^{n-\bar{n}} \times \mathbb{R}^m \to \mathbb{R}^{n-\bar{n}}$$

Suppose that rank $f_b^2|_* = n - \bar{n}$ *so that* $f_b^2|_*$ *is invertible.*

$$\text{Define} \quad \bar{f} : \mathbb{R}^{\bar{n}} \times \mathbb{R}^m \to \mathbb{R}^{\bar{n}}$$
$$\bar{f}(a,\lambda) = f^1(a, h(a,\lambda), \lambda)$$

where h *is the unique smooth function* $U \to \mathbb{R}^{n-\bar{n}}$ *satisfying* $f^2(a,h(a,\lambda),\lambda) = 0$ *and* $h(a_*,\lambda_*) = b_*$ *for some neighborhood* U *of* (a_*,λ_*). *Write* $z = (x,\lambda)$ *and* $\bar{z} = (a,\lambda)$. *Then*

(a) *rank* $f_z = n \iff$ *rank* $\bar{f}_{\bar{z}} = \bar{n}$.
(b) f *has a fold bifurcation at* $z_* = (a_*,b_*,\lambda_*) \iff \bar{f}$ *has a fold bifurcation at* $\bar{z}_* = (a_*,\lambda_*)$.
(c) *For suitable scaling of* $\bar{N}$ *and* $\bar{v}$,

$$N = \bar{N} \text{ and } v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} \bar{v} \\ h_a|_*\bar{v} \end{pmatrix}.$$

Proof (a): Observe that rank f_z = rank $f_{\bar{z},b}$ since f_z and $f_{\bar{z},b}$ differ only in the order of their columns. Note that $\bar{f}_{\bar{z}} = f^1_{\bar{z}} - f^1_b \left(f^2_b\right)^{-1} f^2_{\bar{z}}$ and

$$\begin{pmatrix} f^1_{\bar{z}} & f^1_b \\ f^2_{\bar{z}} & f^2_b \end{pmatrix} \begin{pmatrix} I_{\bar{n}+m} & 0 \\ -\left(f^2_b\right)^{-1} f^2_{\bar{z}} & I_{n-\bar{n}} \end{pmatrix} = \begin{pmatrix} \bar{f}_{\bar{z}} & f^1_b \\ 0 & f^2_b \end{pmatrix} \quad \text{(A6)}$$

The rank of the LHS of (A6) = rank $f_{\bar{z},b}$ since the second matrix is invertible and the rank of the RHS of (A6) = rank $\bar{f}_{\bar{z}}$ + rank f^2_b = rank $\bar{f}_{\bar{z}} + n - \bar{n}$. Hence rank f_z = rank $\bar{f}_{\bar{z}} + n - \bar{n}$ and (a) follows.

(b): By lemma 1(a), f has a fold bifurcation at z_* implies that its zero set Z has a fold at z_* and rank $f_z|_* = n$. In particular, Z is a submanifold of $U_1 \subset \mathbb{R}^n \times \mathbb{R}^m$ where U_1 is a neighborhood of z_*. By part (a), rank $f_{\bar{z}}$ has rank $\bar{n}$ and it follows (see proof of lemma 1(a)) that the zero set $\bar{Z}$ of $\bar{f}$ is a submanifold of $\bar{U}_1 \subset \mathbb{R}^{\bar{n}} \times \mathbb{R}^m$ where $\bar{U}_1$ is a neighborhood of $\bar{z}_*$. Similarly $\bar{f}$ has a fold bifurcation at $\bar{z}_*$ implies that $\bar{Z}$ is a submanifold of $\bar{U}_1$ and Z is a submanifold of U_1. Thus either f has a fold bifurcation at z_* or $\bar{f}$ has a fold bifurcation at $\bar{z}_*$ imply that both Z and $\bar{Z}$ are submanifolds.

$$\text{Define} \quad \zeta : U \to \mathbb{R}^n \times \mathbb{R}^m$$
$$\zeta(a,\lambda) = (a, h(a,\lambda), \lambda)$$
$$\eta : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{\bar{n}} \times \mathbb{R}^m$$
$$\eta(a,b,\lambda) = (a,\lambda)$$

We can find open sets V, $\bar{V}$ with $z_* \in V \subset U_1$, $\bar{z}_* \in \bar{V} \subset \bar{U}_1$ such that

$$\begin{aligned} (a,\lambda) \in \bar{Z} \cap \bar{V} &\iff \bar{f}(a,\lambda) = 0 \\ &\iff f^1(a,h(a,\lambda),\lambda) = 0 \\ &\iff f^1(a,h(a,\lambda),\lambda) = 0 \\ &\quad \text{and } f^2(a,h(a,\lambda),\lambda) = 0 \\ &\iff f(a,h(a,\lambda),\lambda) = 0 \\ &\iff (a,h(a,\lambda),\lambda) \in Z \cap V \end{aligned}$$

Therefore $\zeta|_{\bar{Z}\cap\bar{V}}$ maps $\bar{Z} \cap \bar{V}$ onto $Z \cap V$. Since $\bar{Z} \cap \bar{V}$ is a submanifold of U and $Z \cap V$ is a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ and ζ and η are smooth functions, $\zeta|_{\bar{Z}\cap\bar{V}}$ and $\eta|_{Z\cap V}$ are smooth functions. Moreover, it is easy to verify that $\zeta|_{\bar{Z}\cap\bar{V}} \circ \eta|_{Z\cap V}$ and $\eta|_{Z\cap V} \circ \zeta|_{\bar{Z}\cap\bar{V}}$ are identity functions. Therefore $\zeta|_{\bar{Z}\cap\bar{V}}$ is diffeomorphism from $\bar{Z} \cap \bar{V}$ onto $Z \cap V$. Now use part (a) and lemmas 1(a) and 3 to deduce result (b).

(c): From part (b) and lemmas 1(b) and 3, we obtain

$$N = \bar{N} \text{ and } v = (\zeta|_{\bar{Z}\cap\bar{V}})_{\bar{z}}\Big|_{\bar{z}_*}(\bar{v}) = \begin{pmatrix} \bar{v} \\ h_a|_*\bar{v} \end{pmatrix} \qquad \square$$

Appendix B: Manipulation of a load model

The dynamic load model is specified by

$$\dot{y} = f^1(y,V,\lambda) \quad \text{(B1)}$$
$$0 = g(\dot{P}^d, \dot{V}, P^d, V, \lambda) \quad \text{(B2)}$$
$$P^d = P(y,V) \quad \text{(B3)}$$

We apply lemma 1 by first eliminating P^d and solving g to obtain an expression for $\dot{V}$ to put the equations in state space form. Suppose that $g_{\dot{P}^d} P_V + g_{\dot{V}} \neq 0$. As shown below, this condition is sufficient for equations (B1-B3) to define well posed differential equations in state variables (y,V). Differentiating (B3), substituting in (B2) and using (B1) yields

$$0 = g(P_y f^1(y,V,\lambda) + P_V \dot{V}, \dot{V}, P(y,V), V, \lambda) \quad \text{(B4)}$$

Let (y_*,V_*,λ_*) be an equilibrium of equations (B1-B3). Then (y_*,V_*,λ_*) also satisfies (B4). Since $g_{\dot{P}^d} P_V + g_{\dot{V}} \neq 0$, the implicit function theorem implies that there is an open set $U \ni (y_*,V_*,\lambda_*)$ and a unique smooth function $f^2 : U \to \mathbb{R}$ such that $f^2(y_*,V_*,\lambda_*) = 0$ and

$$0 = g(P_y f^1(y,V,\lambda) + P_V f^2(y,V,\lambda), f^2(y,V,\lambda), P(y,V), V, \lambda) \quad \text{(B5)}$$

Now (B1-B3) can be written in state space form in U:

$$\dot{y} = f^1(y,V,\lambda) \quad \text{(B6)}$$
$$\dot{V} = f^2(y,V,\lambda) \quad \text{(B7)}$$

Applying lemma 2 yields the static equations

$$0 = f^1(y,V,\lambda) \quad \text{(B8)}$$
$$0 = f^2(y,V,\lambda) \quad \text{(B9)}$$

Now we use lemma 4 to deduce that fold bifurcations of equations (B8-9) are the same for the equations

$$0 = f^1(y,V,\lambda) \quad \text{(B10)}$$
$$0 = g(0,0,P(y,V),V,\lambda) \quad \text{(B11)}$$

Equations (B8-B9) imply equations (B10-B11) via (B5). Conversely, equations (B10) and (B5) imply that for $(y,V,\lambda) \in U$,

$$0 = g(P_V f^2(y,V,\lambda), f^2(y,V,\lambda), P(y,V), V, \lambda) \quad \text{(B12)}$$

But f^2 is unique so we can deduce from (B11) and (B12) that $0 = f^2(y,V,\lambda)$. Hence equations (B8-B9) and (B10-B11) are equivalent and have the same zero set Z.

Write $z = (y, V, \lambda)$. To apply lemma 4, it remains to show that the ranks of the Jacobians with respect to z of equations (B8-B9) and (B10-B11) are the same at a point $z_* \in Z$. The Jacobian of (B10-B11) is $\begin{pmatrix} f_z^1|_* \\ a \end{pmatrix}$ where $a = [g(0, 0, P(y, V), V, \lambda)]_z\,|_*$. Differentiating (B5) with respect to z and evaluating at z_* yields

$$0 = \left(g_{\dot{P}^d} P_V + g_{\dot{V}}\right)\Big|_* f_z^2|_* + \left(g_{\dot{P}^d} P_y\right)\Big|_* f_z^1|_* + a$$

and since $g_{\dot{P}^d} P_V + g_{\dot{V}} \neq 0$, we can deduce that rank $\begin{pmatrix} f_z^1|_* \\ a \end{pmatrix} = \text{rank} \begin{pmatrix} f_z^1|_* \\ f_z^2|_* \end{pmatrix}$.

Appendix C: Concerning the definition of fold

This appendix shows that the definition of fold point of Z in Appendix A is standard by proving it equivalent to the definition 4.1 of fold point in [17, Chapter 3 page 87]. The proof also shows a geometric meaning of the transversality condition $_N\pi_{zz}|_*(_V, _V) \neq 0$.

It is convenient to work in a fixed chart of Z with open set $U_1 \ni z_*$ and to neglect the distinction between objects associated with Z and their coordinates. Define the matrix function

$$M : U_1 \to \mathbb{R}^{m\times m}$$
$$z_1 \mapsto \pi_z|_{z_1}$$

and define S_{m-1} to be the matrices in $\mathbb{R}^{m\times m}$ of rank $m-1$. S_{m-1} is a submanifold of $\mathbb{R}^{m\times m}$ (see proof of lemma 6 below or [17, page 60]). If M intersects S_{m-1} transversally at $M(z_*)$, let U_2 be a neighborhood of z_* on which M intersects S_{m-1} transversally and define the submanifold F of U_1 by $F = M^{-1}(S_{m-1}) \cap U_2$.

Then definition 4.1 of [17] may be rewritten and specialized to

$z_* \in U_1$ is a fold point of π iff

(a) $\pi_z|_* = M(z_*) \in S_{m-1}$

(b) M intersects S_{m-1} transversally at $\pi_z|_*$

(c) $TF|_* + \ker \pi_z|_* = TZ|_*$.

Define $\gamma : \mathbb{R} \to Z$ to be a smooth curve in Z with $\gamma(0) = z_*$ and $\gamma_s|_0 = {_V}$, Since $\ker \pi_z|_* = \langle _V \rangle$, condition (c) is equivalent to

(c′) γ intersects F transversally at z_*.

Let W_1 be a neighborhood of 0 in $\mathbb{R}$ so that $\gamma(W_1) \subset U_1$. Define the curve of matrices $A = M \circ \gamma$:

$$A : W_1 \to \mathbb{R}^{m\times m}$$
$$s \mapsto M(\gamma(s))$$

Now we claim that the condition

(d) A intersects S_{m-1} transversally at $\pi_z|_*$

is equivalent to conditions (b) and (c′). For (c′) implies $\gamma_s|_0 \notin TF|_*$ and this, together with (b) implies $M_z(\gamma_s|_0) = A_s|_0 \notin TS_{m-1}|_{M(*)}$ and hence (d). Conversely, (d) implies (b). Moreover, (d) implies (c′) because γ not transverse to F at z_* implies $\gamma_s|_0 \in TF$ implies $A_s|_0 = M_z(\gamma_s|_0) \in TS_{m-1}|_{M(*)}$ implies that A is not transverse to S_{m-1} at $\pi_z|_*$.

Lemma 6 below proves that condition (d) is equivalent to

$$_N A_s|_0 {_V} \neq 0$$

But $A_s|_0 = (M(\gamma(s)))_s\big|_0 = M_z|_* \gamma_s|_0 = \pi_{zz}|_* {_V}$ so condition (d) is equivalent to

$$_N\pi_{zz}|_*(_V, _V) \neq 0,$$

which is the condition ZF(c). But conditions (a) and (b) are equivalent to ZF(a) and ZF(b). Therefore the conditions ZF(a), ZF(b), ZF(c) of the fold definition in Appendix A are equivalent the definition 4.1 of [17].

It remains to state and prove the following:

Lemma 6. *Let $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ be a smooth function so that $A(s)$ is an $m\times m$ matrix parameterized by s. Suppose $A(0)$ has rank $m-1$ and let $_V$ and $_N$ be nonzero vectors such that $A(0){_V} = 0$ and $_N A(0) = 0$. Let S_{m-1} be the submanifold of matrices in $\mathbb{R}^{m\times m}$ with rank $m-1$. Then*

$$A \text{ transversal to } S_{m-1} \text{ at } A(0) \iff {_N}A_s|_0{_V} \neq 0.$$

Proof : Write B^{ij} for the cofactor of the (i, j) element of a matrix $B \in \mathbb{R}^{m\times m}$. Since $A(0)$ has rank $m-1$, i and j can be chosen so that $(A(0))^{ij} \neq 0$. Define the smooth functions $\tilde{_N}(B) = \left(B^{1j}, B^{2j}, \cdots, B^{mj}\right)$ and $\tilde{_V}(B) = \left(B^{i1}, B^{i2}, \cdots, B^{im}\right)^T$. Let $U \subset \mathbb{R}^{m\times m}$ be a neighborhood of $A(0)$ such that $B \in U \Rightarrow B^{ij} \neq 0$. Define

$$\beta :\ U \to \mathbb{R}$$
$$B \mapsto \tilde{_N}(B) B\, \tilde{_V}(B)$$

Write e_i for the column vector with 1 in the ith coordinate and 0 elsewhere. Recall that multiplying elements of a matrix row by their corresponding cofactors and adding yields the matrix determinant while multiplying elements of a matrix row by the cofactors corresponding to another row of the matrix and adding yields zero. Hence

$$B\, \tilde{_V}(B) = \det B\, e_i \ \text{ and } \ \tilde{_N}(B) B = \det B\, e_j^T \tag{C1}$$

and it follows that

$$\beta(B) = B^{ij} \det B$$

Then $S_{m-1} \cap U = \beta^{-1}(0)$ because, in U, $B^{ij} \neq 0$ so that rank $B = m-1$ iff $\det B = 0$. β is regular because the expansion of a determinant in terms of cofactors implies that $[\det B]_{b_{ij}} = B^{ij}$ and hence that $\beta_{b_{ij}} = \left(B^{ij}\right)^2 \neq 0$. (Here $[\det B]_{b_{ij}}$ and $\beta_{b_{ij}}$ denote the gradients of $\det B$ and β with respect to b_{ij}.) $S_{m-1} \cap U = \beta^{-1}(0)$ and β regular imply that $TS_{m-1} = \ker \beta_B$.

Let $V \subset \mathbb{R}$ be a neighborhood of 0 with $A(V) \subset U$. Define $\alpha :\ V \to \mathbb{R}$ by $\alpha(s) = \beta(A(s))$. Then, since $TS_{m-1} = \ker \beta_B$ and β regular, A transversal to S_{m-1} at $A(0) \iff \alpha_s|_0 = \beta_B|_{A(0)} A_s|_0 \neq 0$.

It remains to prove that $\alpha_s|_0 = {_N}A_s|_0{_V}$ for suitable rescalings of $_N$ and $_V$. Equation (C1) implies that $A(0)\tilde{_V}(A(0)) = \det A(0)\, e_i = 0$. Moreover, $\tilde{_V}(A(0)) \neq 0$ since it has a nonzero component $(A(0))^{ij}$. Since $A(0)$ has a one dimensional kernel, a suitable rescaling of $_V$ yields $_V = \tilde{_V}(A(0))$. Similarly $_N = \tilde{_N}(A(0))$. Now

$$\begin{aligned}\alpha_s|_0 &= \left[\tilde{_N}(A(s))\, A(s) \tilde{_V}(A(s))\right]_s\Big|_0 \\ &= \left[\tilde{_N}(A(s))\right]_s|_0 A(0){_V} + {_N}A_s|_0{_V} \\ &\quad + {_N}A(0)\left[\tilde{_V}(A(s))\right]_s|_0 \\ &= {_N}A_s|_0{_V}\end{aligned}$$

□