Transform Inversion by Partial Fractions

Suppose that

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}}, \quad q < p.$$

We first point out that the denominator has exactly p roots. Consider the polynomial

$$\Xi(z) := 1 + a_1 z + \cdots + a_p z^p.$$

Since $\Xi(1/z) = A(z)$, we see that $A(z) = 0$ if and only if $\Xi(1/z) = 0$. Since Ξ is a polynomial of degree p, it has exactly p roots, say $\xi_1, \ldots, \xi_p$. Furthermore, since $\Xi(0) = 1$, none of the ξ_k is zero. Hence, the p roots of $A(z)$ are $\alpha_1 := 1/\xi_1, \ldots, \alpha_p := 1/\xi_p$.

We now show that if the denominator roots α_k are distinct, then

$$H(z) = \sum_{k=1}^{p} \frac{C_k}{1 - \alpha_k z^{-1}}.$$

Suppose that such a formula for $H(z)$ exists. Write it as

$$\frac{B(z)}{\prod_{k=1}^{p}(1 - \alpha_k z^{-1})} = \sum_{k=1}^{p} \frac{C_k}{1 - \alpha_k z^{-1}}.$$

Multiply both sides by $1 - \alpha_i z^{-1}$ to get

$$\frac{B(z)}{\prod_{k \neq i}^{p}(1 - \alpha_k z^{-1})} = C_i + \sum_{k \neq i} \frac{C_k(1 - \alpha_i z^{-1})}{1 - \alpha_k z^{-1}}.$$

In this expression, now set $z = \alpha_i$ to get

$$\frac{B(\alpha_i)}{\prod_{k \neq i}^{p}(1 - \alpha_k/\alpha_i)} = C_i. \tag{1}$$

Example. Find the partial fraction expansion of

$$H(z) = \frac{1}{1 - 7z^{-1} + 12z^{-2}}.$$

Solution. Writing

$$1 - 7z^{-1} + 12z^{-2} = z^{-2}(z^2 - 7z + 12) = z^{-2}(z-3)(z-4),$$

we see that the **poles** (roots of the denominator) are $z = 3$ and $z = 4$. Write

$$\frac{1}{(1 - 3z^{-1})(1 - 4z^{-1})} = \frac{?}{1 - 3z^{-1}} + \frac{?}{1 - 4z^{-1}}.$$

The first missing numerator is

$$\left.\frac{1}{1-4z^{-1}}\right|_{z=3} = \frac{1}{1-4/3} = -3$$

and the second missing numerator is

$$\left.\frac{1}{1-3z^{-1}}\right|_{z=4} = \frac{1}{1-3/4} = 4.$$

Hence,

$$\frac{1}{(1-3z^{-1})(1-4z^{-1})} = \frac{-3}{1-3z^{-1}} + \frac{4}{1-4z^{-1}}.$$

In many cases, the degrees of $A(z)$ and $B(z)$ are the same. In this case, observe that

$$\begin{aligned}
\frac{B(z)}{A(z)} &= \frac{\left[B(z)-\frac{b_p}{a_p}A(z)\right]+\frac{b_p}{a_p}A(z)}{A(z)} \\
&= \frac{\left[\{b_p z^{-p}+\cdots\}-\frac{b_p}{a_p}\{a_p z^{-p}+\cdots\}\right]+\frac{b_p}{a_p}A(z)}{A(z)} \\
&= \frac{\widetilde{B}(z)}{A(z)}+\frac{b_p}{a_p},
\end{aligned}$$

where $\widetilde{B}(z) = B(z) - \frac{b_p}{a_p}A(z)$ is a polynomial of degree $q < p$.

Example. Find the partial fraction expansion of

$$\frac{2z^{-1}-z^{-2}}{1-5z^{-1}+6z^{-2}}.$$

Solution. Write

$$\begin{aligned}
\frac{2z^{-1}-z^{-2}}{1-5z^{-1}+6z^{-2}} &= \frac{2z^{-1}-z^{-2}+(1/6)(1-5z^{-1}+6z^{-2})}{1-5z^{-1}+6z^{-2}} - \frac{1}{6} \\
&= \frac{2z^{-1}-z^{-2}+(1/6)-(5/6)z^{-1}+z^{-2}}{1-5z^{-1}+6z^{-2}} - \frac{1}{6} \\
&= \frac{(7/6)z^{-1}+(1/6)}{1-5z^{-1}+6z^{-2}} - \frac{1}{6} \\
&= \frac{-3/2}{1-2z^{-1}} + \frac{5/3}{1-3z^{-1}} - \frac{1}{6}.
\end{aligned}$$

Problems for HW #9

1. Find the partial fraction expansion of
$$\frac{6z^{-1}}{1-6z^{-1}+11z^{-2}-6z^{-3}}.$$
Answer (numerators): 3, -12, 9.

2. Find the partial fraction expansion of
$$\frac{2+z^{-1}}{1-12z^{-1}+35z^{-2}}.$$
Answer (numerators): $15/2$, $-11/2$.

3. Find the partial fraction expansion of
$$\frac{(17/6)z^{-1}-(1/6)}{1-5z^{-1}+6z^{-2}}.$$
Answer (numerators): $-5/2$, $7/3$.

4. Find the partial fraction expansion of
$$\frac{4-23z^{-1}+70z^{-2}}{1-12z^{-1}+35z^{-2}}.$$
Answer (numerators): $-11/2$, $15/2$.

5. Find the partial fraction expansion of
$$\frac{z^{-2}(1+z^{-1})}{1-6z^{-1}+11z^{-2}-6z^{-3}}.$$
Answer (numerators): 1, $-3/2$, $2/3$.

6. **MATLAB.** Write a MATLAB script that takes as input $\mathtt{a}=[1,a_1,\ldots,a_p]$ and $\mathtt{b}=[b_0,\ldots,b_q]$ for $q<p$ and finds the coefficients C_i in (1). Be sure to print out the root that C_i corresponds to. To find the poles, use the MATLAB function `roots`. To evaluate polynomials from knowledge of their coefficients, use the MATLAB function `polyval`. To this end, the MATLAB function `fliplr` will also be helpful. To evaluate products, the MATLAB function `prod` can be used. We also mention that the MATLAB function `rat` or the command `format rat` may be useful to express answers as fractions. **Test your script on each of the preceding problems.**

7. Let $m(t)$ be a message signal of bandwidth f_c. In a double-sideband suppressed-carrier (DSB-SC) system, the message $m(t)$ is used to modulate a sinusoidal carrier, say $\cos(2\pi f_0 t)$, to generate the waveform $x(t) := m(t)\cos(2\pi f_0 t)$, which is transmitted to a receiver. At the receiver, the

waveform $x(t)$ is multiplied by $\cos(2\pi f_0 t)$ to produce $y(t) := x(t)\cos(2\pi f_0 t)$.

(a) The modulation property of Fourier transforms says that if $x(t) = m(t)\cos(2\pi f_0 t)$, then

$$X(f) = \frac{1}{2}[M(f - f_0) + M(f + f_0)].$$

If $m(t) = 2f_c \operatorname{sinc}(2f_c t)$, and $f_0 > f_c$, sketch $X(f)$.

(b) If $y(t) = x(t)\cos(2\pi f_0 t)$, sketch $Y(f)$ using $X(f)$ from part (a).

(c) Among the signals $m(t)$, $x(t)$, and $y(t)$, $y(t)$ has the highest bandwidth. What is the Nyquist rate for alias-free sampling of $y(t)$?

8. **MATLAB.** It should be clear from the preceding problem that if $y(t)$ is applied to an ideal lowpass filter (LPF) of gain 2 and bandwidth anything between f_c and $2f_0 - f_c$, then the LPF output is exactly the message waveform $m(t)$. This observation holds for any message waveform of bandwidth at most f_c. In this problem you will do the lowpass filtering with a discrete-time system.

- Let the message bandwidth be approximately f_c, and let the observation time window be $0 \le t \le T$, where $T := 10/f_c$. To make the carrier frequency greater than the message bandwidth, take $f_0 := 10f_c$. In this problem, all signals are time limited, and so the signals are not bandlimited. But they are approximately bandlimited. So, for the sampling rate f_{samp}, use 1.2 times the Nyquist-rate formula of the preceding problem.
- Let $m(t) = 2f_c \operatorname{sinc}(2f_c[t - T/2])$, $x(t) = m(t)\cos(2\pi f_0 t)$, and $y(t) = x(t)\cos(2\pi f_0 t)$, all for $0 \le t \le T$.
- Plot $m(t)$, $|M(f)|$, $x(t)$, and $|X(f)|$. These are continuous-time Fourier transforms, but you can approximate them with the DTFT of the samples. For this, you may use `dtftfft` with $m(t)$ and $x(t)$ sampled at the same sampling times used for $y(t)$. Be sure the horizontal and vertical axes are properly scaled for CTFTs.
- Design a discrete-time LPF using the bilinear transformation of a Butterworth filter. If $H(f)$ denotes the DTFT of the filter, plot $20\log_{10}|H(f_{\text{samp}} f)|$ as a function of $f_{\text{fsamp}} f$ for $|f| \le 1/2$. Also design a discrete-time LPF by applying a Kaiser window to an ideal LPF with generalized linear phase (GLP). Plot its frequency response too.
- Apply samples of $y(t)$ from $0 \le t \le T$ to the IIR LPF and plot the resulting output samples; on the same graph also plot the original $m(t)$ for $0 \le t \le T$. Repeat for the FIR filter.