J

Harmonic Entropy

Harmonic entropy is a measure of the uncertainty in pitch perception, and it provides a physical correlate of tonalness, one aspect of the psychoacoustic concept of dissonance. This Appendix shows in detail how to calculate harmonic entropy and continues the discussion in Sect. 5.3.3.

Harmonic entropy was introduced by Erlich [W: 9] as a refinement of a model by van Eck [B: 125]. It is based on Terhardt's [B: 196] theory of harmony, and it follows in the tradition of Rameau's fundamental bass [B: 145]. It provides a way to measure the uncertainty of the fit of a harmonic template to a complex sound spectrum. As a major component of tonalness is the closeness of the partials of a complex sound to a harmonic series, high tonalness corresponds to low entropy and low tonalness corresponds to high entropy.

In the simplest case, consider two harmonic tones. If the tones are to be understood as approximate harmonic overtones of some common root, they must form a simple-integer ratio with one another. One way to model this uses the Farey series $\mathcal{F}_n$ of order n, which lists all ratios of integers up to n. For example, $\mathcal{F}_6$ is

$$\frac{0}{1}, \quad \frac{1}{6}, \quad \frac{1}{5}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{2}{5}, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{5}{6}, \quad \frac{1}{1}.$$

A useful property of the Farey series is that the distance between successive terms is larger when the ratios are simpler. Let the jth element of the series be $f_j = \frac{a_j}{b_j}$. Then the region over which f_j dominates goes from the mediant[1] below to the mediant above, that is, from $\frac{a_{j-1}+a_j}{b_{j-1}+b_j}$ to $\frac{a_j+a_{j+1}}{b_j+b_{j+1}}$. Designate this region r_j. Figure J.1 plots the length of r_j vs. f_j for $\mathcal{F}_{50}$, the Farey series of order 50. Observe that complex ratios cluster together, and that the simple ratios tend to separate. Thus, simple ratios like 1/2, 2/3, and 3/4 have wide regions with large r_j, and complex ratios tend to have small regions with small r_j.

For any interval i, a Gaussian distribution (a bell curve) is used to associate a probability $p_j(i)$ with the ratio f_j in $\mathcal{F}_n$. The probability that interval i is perceived as a mistuning of the jth member of the Farey series is

$$p_j(i) = \frac{1}{\sigma\sqrt{2\pi}} \int_{t \in r_j} e^{-(t-i)^2/2\sigma^2} dt.$$

[1] Recall that the mediant of two ratios $\frac{a}{b}$ and $\frac{c}{d}$ is the fraction $\frac{a+c}{b+d}$.

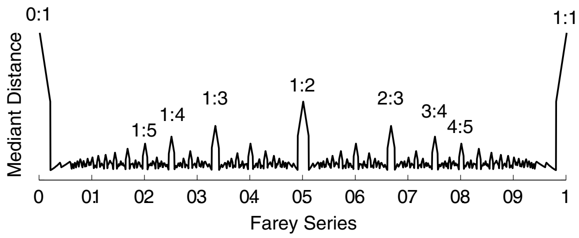

Fig. J.1. The mediant distances between entries (the length of the r_j) are plotted as a function of the small integer ratios f_j drawn from the Farey series of order 50. The simplest ratios dominate.

Thus, the probability is high when the i is close to f_j and low when i is far from f_j. This is depicted in Fig. J.2 where the probabilities that i is perceived as f_{j+1}, f_{j+2}, and f_{j+3} are shown as the three regions under the bell curve. Erlich refines this model to incorporate the $\log$ of the intervals and mediants, which is sensible because pitch perception is itself (roughly) logarithmic.

The harmonic entropy (HE) of i is then defined (parallel to the definition of entropy used in information theory) as

$$HE(i) = -\sum_j p_j(i) \log(p_j(i)).$$

When the interval i lies near a simple-integer ratio f_j, there will be one large probability and many small ones. Harmonic entropy is low. When the interval i is distant from any simple-integer ratio, many complex ratios contribute many nonzero probabilities. Harmonic entropy is high. A plot of harmonic entropy over an octave of intervals i (labeled in cents) appears in Fig. 5.5 on p. 89. This figure used $\mathcal{F}_{50}$ and $\sigma = 0.007$. Clearly, intervals that are close to simple ratios are distinguished by having low entropy, and more complex intervals have high harmonic entropy.

Generalizations of the harmonic entropy measure to consider more than two sounds at a time are currently under investigation; one possibility involves Voronoi cells. Harmonic series triads with simple ratios are associated with large Voronoi cells, whereas triads with complex ratios are associated with small cells. This nicely parallels the dyadic case. Recall the example (from p. 96 and sound examples [S: 40]–[S: 42]), which compares the clusters 4:5:6:7 with 1/7:1/6:1/5:1/4. In such cases, the harmonic entropy model tends to agree better with listener's perceptions of the dissonance of these chords than does the sensory dissonance approach. Paul Erlich comments that the study of harmonic entropy is a "public work in progress" at [W: 9].

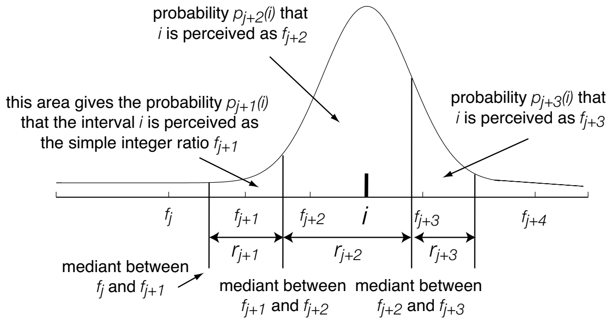

Fig. J.2. Each region r_{j+1} extends from the mediant between f_j and f_{j+1} to the mediant between f_{j+1} and f_{j+2}. The interval i specifies the mean of the Gaussian curve, and the probabilities $p_j(i)$ are defined as the disjoint areas between the axis and the curve.